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基于有限差分-谱方法的分数阶 Burgers 流体的非稳态驻点流动

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摘要:研究了分数阶 Burgers 流体通过拉伸平板的非稳态驻点流动问题。将分数阶导数引入 Burgers 流体模型可以更好地模拟流动过程,但也增加了模型的复杂性和求解难度。首次运用有限差分-谱方法求解分数阶 Burgers 流体模型,离散格式构造简单有效。采用谱方法对控制方程中的空间项进行离散,利用有限差分方法分别结合 L-1 和 L-2 算法离散控制方程中的时间项,给出了两种离散格式,并且通过构造数值算例证明了离散格式的收敛性。结果表明,在靠近平板处,速度随着分数阶导数的增加而减小,而无穷远处的流体速度呈现出相反的趋势,体现了分数阶导数的记忆特性。此外,雷诺数越小,流体的粘度越大,导致流体速度越大。由于松弛时间参数的松弛特性,靠近平板处松弛时间参数对速度分布有抑制作用,远离平板处松弛时间促进流体流动。

关键词:非稳态;驻点流;分数阶 Burgers 流体;L-2 算法;有限差分-谱方法

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1 引言

工业生产中粘弹性流体在特定流动条件下的数值预测具有重要的现实意义,其提供了一种有效的方法来研究实际聚合物加工操作的复杂流动机制。塑料在模具中的挤出过程是驻点流动的广泛应用之一,已有许多文章致力于各种粘弹性流体驻点流动问题的数值研究。Labropulu 等^[1]对二级流体的非稳态驻点流动进行了研究。Ahmed 等^[2]讨论了 Maxwell 流体非稳态驻点流中的混合对流。Bai 等^[3]探究了上对流 Oldroyd-B 纳米流体的非稳态驻点流动及弛豫-延迟双扩散模型。Burgers 流体模型是由 Maxwell 单元和 Kelvin 单元串联而成,该模型已用于表征奶酪^[4]、土壤^[5]和沥青^[6]等粘弹性材料,因此,将 Burgers 流体模型应用到非稳态驻点流动的研究具有重要的现实意义。作为一个四参量模型,公式推导复杂,目前 Burgers 流体的非稳态驻点流动研究很少,且都为整数阶导数模型。

近年来,将分数阶导数引入粘弹性流体的本构方程已证明是处理粘弹性流体性质的一种有价值的工具^[7]。一般来说,这些方程是通过从已知模型中用分数阶导数替换整数阶导数,相较于整数阶 Burgers 流体本构方程,分数阶 Burgers 流体模型与早龄期混凝土徐变过程的实验数据保持更高的一致性^[8]。因此,很多研究致力于用分数阶微积分方法讨论 Burgers 流体的流动问题。Liu 等^[9]分析了指数加速板引起的分数阶 Burgers 流体在磁场作用下的流动。Han 等^[10]研究了分数阶 Burgers 流体在两侧壁间流动的滑移效应。Abdulhameed 等^[11]提出了分数阶 Burgers 流体通过微通道的电渗流动模型。目前没有驻点流的研究涉及分数阶 Burgers 流体本构方程,本文首次将分数阶 Burgers 流体模型应用到非稳态驻点流动。

数值求解描述流体运动的偏微分方程已成为流体力学理论研究的重要手段^[12-13]。有限差分方法用差商代替微商,格式构造方便灵活,易于模拟各种物理性质。谱方法的思想是利用特殊的正交

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多项式(如 Chebyshev 多项式)作为试探函数来离散偏微分方程,其特点是在较少未知量的情况下,能产生高精度的逼近。因此,有限差分-谱方法已广泛应用于有限域流体流动。Li 等^[14]采用时间有限差分/空间谱方法对分数阶 Nernst-Planck 方程进行数值求解。Li 等^[15]利用一阶有限差分-谱方法求解 Allen-Cahn 方程。Yu 等^[16]基于时间方向上的有限差分法和空间方向上的谱方法,求解了二维时间 Caputo-Fabrizio 分数阶扩散方程。然而,目前还没有研究用有限差分-谱方法求解分数阶 Burgers 流体模型,L-1 算法和 L-2 算法可以有效解决分数阶 Burgers 流体模型中由于分数阶导数参数的取值造成离散过程复杂的问题,并且采用切比雪夫谱方法,可以避免吉布斯现象和龙格现象,因此,本文应用有限差分-谱方法作为求解方法。

基于上述讨论,本文首次将分数阶 Burgers 流体模型应用到驻点流动中,结合 L-1 算法和 L-2 算法给出了控制方程的两种离散格式,通过构造数值算例验证了离散格式是收敛的,首次用有限差分-谱方法得到分数阶 Burgers 流体模型的数值解。最后,通过图像变化分析了不同参数对速度分布的影响。

2 数学模型

考虑分数阶 Burgers 流体在拉伸板上的二维流动问题,如图 1 所示,流体垂直流向平板,撞击平板时产生速度为 0 的点,即为驻点,而后流体沿拉伸平板分流。本文假设沿着平板的方向为 x 轴,垂直于平板的方向为 y 轴,建立坐标系。 u_∞ 和 v_∞ 分别表示无穷远处流体的速度在 x 方向和 y 方向的分量, u_w 表示平板的拉伸速度。

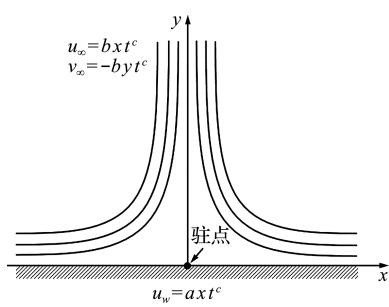


图 1 物理模型
Fig. 1 Diagram of physical model

分数阶 Burgers 流体的本构方程为^[17]

$$(1 + \lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha}) \mathbf{S} = \mu (1 + \lambda_3^{\beta} D_t^{\beta}) \mathbf{A} \quad (1)$$

式中 \mathbf{S} 为剪切应力张量, \mathbf{A} 为 Rivlin-Ericksen 张量, λ_1 和 λ_3 分别为松弛时间和延迟时间, λ_2 为量

纲为 $[T^2]$ 的材料常数, μ 为流体的粘度系数, D_t^{α} 为 α 阶的 Caputo 型分数阶导数算子, 其定义为^[20]

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\eta)^{-\alpha} \frac{\partial f(\eta)}{\partial \eta} d\eta \quad (2)$$

式中 $\Gamma(\cdot)$ 为伽马函数。与上述定义类似, $D_t^{2\alpha}$ 和 D_t^{β} 分别表示 2α 阶和 β 阶的 Caputo 型分数阶导数算子, $0 < \alpha < 1, 0 < \beta < 1$ 。

动量方程^[22]为

$$\rho(d\mathbf{V}/dt) = -\nabla p + \nabla \cdot \mathbf{S} \quad (3)$$

式中 $\mathbf{V} = (u, v)$ 为流体的速度, u 和 v 分别为流体在 x 和 y 方向的速度分量, ρ 为流体的密度, p 为压强。

将式(1)代入式(3)并进行边界层近似, 得到边界层内的动量方程为

$$(1 + \lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha}) \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(1 + \lambda_3^{\beta} \frac{\partial^{\beta}}{\partial t^{\beta}} \right) \frac{\partial^2 u}{\partial y^2} \quad (4)$$

式中 $\nu = \mu/\rho$ 为流体的运动粘性系数, 结合无穷远处的速度边界条件可以得到无穷远处的动量方程, 进而可以得到无穷远处压强梯度的表达式, 由于无穷远处的压强与边界层内的压强相等, 即 $p_{\infty} = p$, 则可得到边界层内的压强梯度表达式, 将其代入边界层内的动量方程(4), 所以, 分数阶 Burgers 流体的流动连续性方程和动量控制方程为

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (5)$$

$$(1 + \lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha}) \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \nu \left(1 + \lambda_3^{\beta} D_t^{\beta} \right) \frac{\partial^2 u}{\partial y^2} + (1 + \lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha}) \times \left(\frac{\partial u_{\infty}}{\partial t} + u_{\infty} \frac{\partial u_{\infty}}{\partial x} \right) \quad (6)$$

边界层控制方程相应的初始条件和边界条件为

$$t = 0, u = 0, v = 0 \quad (7)$$

$$t > 0, u|_{y=0} = u_w = axt^c \quad (8)$$

$$u|_{y \rightarrow \infty} = u_{\infty} = bxt^c, v|_{y \rightarrow \infty} = v_{\infty} = -byt^c \quad (9)$$

式中 a 和 b 表示量纲为 $[T^{-c-1}]$ 的常数, c 为幂指数参数。

引入以下无量纲变量对式(5~9)进行无量纲化

$$u^* = \frac{u}{aL_0 t_0^c}, v^* = \frac{v}{aL_0 t_0^c}, u_w^* = \frac{u_w}{aL_0 t_0^c}, t^* = \frac{t}{t_0},$$

$$\lambda_1^* = \frac{\lambda_1}{t_0}, u_{\infty}^* = \frac{u_{\infty}}{aL_0 t_0^c}, \lambda_2^* = \frac{\lambda_2}{t_0^2}, \lambda_3^* = \frac{\lambda_3}{t_0}, x^* = \frac{x}{L_0},$$

$$y^* = y/L_0, Re = \rho u_w L_0 / \mu, Fu = at_0^{c+1}$$

式中 L_0 和 t_0 分别表示特征长度和特征时间, Re 称为雷诺数, Fu 表示一个与时间相关的无量纲数, 称为动量傅里叶数^[19]。此外, 令 $u_0 = aL_0 t_0^c$ 。因

此,无量纲的边界层控制方程为(为了表达简便,此处省略*)

$$\partial u / \partial x + \partial v / \partial y = 0 \quad (10)$$

$$(1 + \lambda_1^\alpha D_t^\alpha + \lambda_2^\alpha D_t^{2\alpha}) \left(\frac{1}{F_u} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \frac{1}{Re} (1 + \lambda_3^\beta D_t^\beta) \frac{\partial^2 u}{\partial y^2} + (1 + \lambda_1^\alpha D_t^\alpha + \lambda_2^\alpha D_t^{2\alpha}) \times \left(\frac{1}{F_u} \frac{\partial u_\infty}{\partial t} + u_\infty \frac{\partial u_\infty}{\partial x} \right) \quad (11)$$

无量纲的初始条件和边界条件为

$$t = 0, u = 0, v = 0 \quad (12)$$

$$t > 0, u|_{y=0} = u_w = xt^c \quad (13)$$

$$u|_{y \rightarrow \infty} = u_\infty = sx t^c, v|_{y \rightarrow \infty} = v_\infty = -sy t^c \quad (14)$$

式中 $s = b/a$ 称为驻点参数。

3 有限差分-谱方法

利用有限差分-谱方法对分数阶 Burgers 流体边界层控制方程组(10~14)进行数值求解^[20]。

对于时间项离散,采用有限差分方法结合 L-1 算法和 L-2 算法对具有多项时间分数阶导数的控制方程进行了数值离散。整数阶时间项在网格点 $(x_i, y_j, t_{k-1/2})$ 处的离散格式为

$$\partial u / \partial t \approx (u_{i,j}^k - u_{i,j}^{k-1}) / \Delta t \quad (15)$$

式中 Δt 为时间步长, u 在网格点 (x_i, y_j, t_k) 上的数值解表示为 $u_{i,j}^k$ 。

当 $0 < \alpha < 1$ 时,采用 L-1 算法进行数值离散, Caputo 型分数阶导数项在网格点 $(x_i, y_j, t_{k-1/2})$ 处的离散格式为^[17]

$$D_t^\alpha \frac{\partial u}{\partial t} \approx \frac{\Delta t^{-\alpha-1}}{\Gamma(2-\alpha)} [(u_{i,j}^k - u_{i,j}^{k-1}) - \alpha_{k-1}^{(\alpha)} (u_{i,j}^1 - u_{i,j}^0)] - \frac{\Delta t^{-\alpha-1}}{\Gamma(2-\alpha)} \sum_{n=1}^{k-1} (\alpha_{k-n-1}^{(\alpha)} - \alpha_{k-n}^{(\alpha)}) (u_{i,j}^n - u_{i,j}^{n-1}) \quad (16)$$

式中 $\alpha_n^{(\alpha)} = (n+1)^{1-\alpha} - n^{1-\alpha}$ ($n = 0, 1, 2, \dots, F_1$)。

当 $1 < \alpha < 2$ 时,采用 L-2 算法进行数值离散, Caputo 型分数阶导数项在网格点 $(x_i, y_j, t_{k-1/2})$ 处的离散格式为^[21]

$$D_t^\alpha \frac{\partial u}{\partial t} \approx \frac{\Delta t^{-\alpha-1}}{\Gamma(3-\alpha)} (u_{i,j}^k - u_{i,j}^{k-1}) - \frac{\Delta t^{-\alpha-1}}{\Gamma(3-\alpha)} \times (u_{i,j}^{k-1} - u_{i,j}^{k-2}) - \frac{\Delta t^{-\alpha-1}}{\Gamma(3-\alpha)} \sum_{n=1}^{k-1} (b_{k-n-1}^{(\alpha)} - b_{k-n}^{(\alpha)}) \times (u_{i,j}^n - u_{i,j}^{n-1}) + \frac{\Delta t^{-\alpha-1}}{\Gamma(3-\alpha)} \sum_{n=1}^{k-2} (b_{k-n-2}^{(\alpha)} - b_{k-n-1}^{(\alpha)}) \times (u_{i,j}^n - u_{i,j}^{n-1}) \quad (17)$$

式中 $b_n^{(\alpha)} = (n+1)^{2-\alpha} - n^{2-\alpha}$ ($n = 0, 1, 2, \dots, F_2$)。

对于控制方程中的空间项,采用谱方法进行数值离散。以拉格朗日多项式作为基函数, (ξ_i, ξ_j) 为切比雪夫点, $\xi_i = \cos(i\pi/N)$ ($i = 0, 1, \dots, N$)。首

先令 $x_i = (\xi_i + 1)(X/2)$, $y_j = (\xi_j + 1)(Y/2)$ ($j = 0, 1, \dots, N$), 将物理区域转化为有限域 $[0, X] \times [0, Y]$, 网格点为 (x_i, y_j) 。在固定时间,有

$$u(x, y, t_k) \approx \sum_{p=0}^N \sum_{q=0}^N u_{p,q}^k L_p(x) L_q(y) \quad (18)$$

因此,在 $t = t_{k-1/2}$ 时

$$u(x, y, t_{k-1/2}) \approx \sum_{p=0}^N \sum_{q=0}^N \frac{u_{p,q}^k + u_{p,q}^{k-1}}{2} L_p(x) L_q(y) \quad (19)$$

式中 $L_p(x) = \prod_{k=0, k \neq p}^N (x - x_k) / (x_p - x_k)$, $L_q(y) = \prod_{m=0, m \neq q}^N (y - y_m) / (y_q - y_m)$ 为拉格朗日多项式, 基于切比雪夫点的一阶导数微分矩阵 \mathbf{D}_N 为

$$(\mathbf{D}_N)_{i,j} = \begin{cases} \frac{2N^2 + 1}{6} & (i = j = 0) \\ \frac{c_i}{c_j} \frac{(-1)^{i+j}}{\xi_i - \xi_j} & (i \neq j, i, j = 1, 2, \dots, N-1) \\ \frac{-\xi_j}{2(1 - \xi_j^2)} & (i = j = 1, 2, \dots, N-1) \\ -\frac{2N^2 + 1}{6} & (i = j = N) \end{cases} \quad (20)$$

式中 $c_i = \begin{cases} 2 & (i = 0 \text{ 或 } N) \\ 1 & (\text{其他}) \end{cases}$ 。二阶微分矩阵 $\mathbf{D}_N^{(2)}$

可以由 \mathbf{D}_N 计算得到, 即 $\mathbf{D}_N^{(2)} = \mathbf{D}_N^2$ 。将张量积 \otimes 引入二维空间项的离散格式中^[24]。矩阵 \mathbf{H} 和 \mathbf{M} 的张量积记为 $\mathbf{H} \otimes \mathbf{M}$, 如果 \mathbf{H} 和 \mathbf{M} 分别是 $m \times n$ 和 $r \times s$ 维的矩阵, 则 $\mathbf{H} \otimes \mathbf{M}$ 为 $mr \times ns$ 的矩阵

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & 2m_{11} & 2m_{12} \\ m_{21} & m_{22} & 2m_{21} & 2m_{22} \\ 3m_{11} & 3m_{12} & 4m_{11} & 4m_{12} \\ 3m_{21} & 3m_{22} & 4m_{21} & 4m_{22} \end{bmatrix}$$

\mathbf{D}_N 的表达式在 $[-1, 1]$ 上获得, 因此, 在对其他区间上的函数求导之前需要将微分矩阵 \mathbf{D}_N 缩放, 即 $\mathbf{D}_X = (2/X) * \mathbf{D}_N$, $\mathbf{D}_Y = (2/Y) * \mathbf{D}_N$ ^[22]。控制方程中空间项的离散格式为

$$\frac{\partial u}{\partial x} \approx \sum_{q=0}^N \sum_{p=0}^N (\mathbf{E} \otimes \mathbf{D}_X)_{j(N+1)+i+1, q(N+1)+p+1} \frac{u_{p,q}^k + u_{p,q}^{k-1}}{2} \quad (21)$$

$$\frac{\partial u}{\partial y} \approx \sum_{q=0}^N \sum_{p=0}^N (\mathbf{D}_Y \otimes \mathbf{E})_{j(N+1)+i+1, q(N+1)+p+1} \frac{u_{p,q}^k + u_{p,q}^{k-1}}{2} \quad (22)$$

$$\frac{\partial^2 u}{\partial y^2} \approx \sum_{q=0}^N \sum_{p=0}^N (\mathbf{D}_Y^2 \otimes \mathbf{E})_{j(N+1)+i+1, q(N+1)+p+1} \frac{u_{p,q}^k + u_{p,q}^{k-1}}{2} \quad (23)$$

式中 \mathbf{E} 为 $N+1$ 维的单位矩阵。为了书写方便, 用下列的记号表示矩阵的张量积。

$$\mathbf{Z} = \mathbf{E} \otimes \mathbf{D}_X, \quad \mathbf{P} = \mathbf{D}_Y \otimes \mathbf{E}, \quad \mathbf{Q} = \mathbf{D}_Y^2 \otimes \mathbf{E}$$

此外, 控制方程中的 v 也有类似的规定。

因此, 根据上述离散格式, 方程(5)的离散格式为

$$\sum_{q=0}^N \sum_{p=0}^N \mathbf{Z}_{j(N+1)+i+1, q(N+1)+p+1} \frac{u_{p,q}^k + u_{p,q}^{k-1}}{2} + \sum_{q=0}^N \sum_{p=0}^N \mathbf{P}_{j(N+1)+i+1, q(N+1)+p+1} \frac{v_{p,q}^k + v_{p,q}^{k-1}}{2} = 0 \quad (24)$$

当 $0 < \alpha < 0.5, 0 < \beta < 1$, 即 $0 < 2\alpha < 1, 0 < \beta < 1$ 时, 方程(6)的离散格式为

$$\begin{aligned} & \sum_{q=0}^N \sum_{p=0}^N (\mathbf{G}_1)_{j(N+1)+i+1, q(N+1)+p+1} u_{p,q}^k = \\ & r_{16} x_i [(k-1)t]^{c-1} + s^2 x_i [(k-1)t]^{2c} + \\ & r_{17} \lambda_1^\alpha x_i [(k-1)t]^{c-\alpha-1} + r_{18} \lambda_1^\alpha x_i [(k-1)t]^{2c-\alpha} + \\ & r_{12} (A_1)_{i,j}^k + r_{14} (A_2)_{i,j}^k + r_{17} \lambda_2^\alpha x_i [(k-1)t]^{c-2\alpha-1} + \\ & r_{18} \lambda_2^\alpha x_i [(k-1)t]^{2c-2\alpha} - \left(\frac{1}{2} + r_{13} + r_{15} \right) \times \\ & u_{i,j}^{k-1} \sum_{q=0}^N \sum_{p=0}^N (\mathbf{Z}_{j(N+1)+i+1, q(N+1)+p+1}) u_{p,q}^{k-1} - \left(\frac{1}{2} + \right. \\ & \left. r_{13} + r_{15} \right) u_{i,j}^{k-1} \sum_{q=0}^N \sum_{p=0}^N (\mathbf{P}_{j(N+1)+i+1, q(N+1)+p+1}) u_{p,q}^{k-1} + \\ & \left(\frac{1}{2\text{Re}} + r_{11} \right) \sum_{q=0}^N \sum_{p=0}^N (\mathbf{Q}_{j(N+1)+i+1, q(N+1)+p+1}) u_{p,q}^{k-1} + \\ & \left(\frac{1}{\Delta t} + r_{12} + r_{14} \right) \sum_{q=0}^N \sum_{p=0}^N (\mathbf{E}_{j(N+1)+i+1, q(N+1)+p+1}) \times \\ & u_{p,q}^{k-1} - r_{11} \sum_{q=0}^N \sum_{p=0}^N (\mathbf{Q}_{j(N+1)+i+1, q(N+1)+p+1}) \times \\ & \sum_{n=1}^{k-1} (\alpha_{k-n-1}^{(\beta)} - \alpha_{k-n}^{(\beta)}) (u_{p,q}^n + u_{p,q}^{n-1}) - \\ & r_{11} \alpha_{k-1}^{(\beta)} \sum_{q=0}^N \sum_{p=0}^N (\mathbf{Q}_{j(N+1)+i+1, q(N+1)+p+1}) \times \\ & (u_{p,q}^1 + u_{p,q}^0) + r_{13} \sum_{n=1}^{k-1} (\alpha_{k-n-1}^{(\alpha)} - \alpha_{k-n}^{(\alpha)}) \times \\ & u_{i,j}^{n-1} \sum_{q=0}^N \sum_{p=0}^N (\mathbf{Z}_{j(N+1)+i+1, q(N+1)+p+1}) \times \\ & (u_{p,q}^n + u_{p,q}^{n-1}) + r_{13} \sum_{n=1}^{k-1} (\alpha_{k-n-1}^{(\alpha)} - \alpha_{k-n}^{(\alpha)}) \times \\ & u_{i,j}^{n-1} \sum_{q=0}^N \sum_{p=0}^N (\mathbf{P}_{j(N+1)+i+1, q(N+1)+p+1}) \times \\ & (u_{p,q}^n + u_{p,q}^{n-1}) + r_{15} \sum_{n=1}^{k-1} (\alpha_{k-n-1}^{(2\alpha)} - \alpha_{k-n}^{(2\alpha)}) \times \\ & u_{i,j}^{n-1} \sum_{q=0}^N \sum_{p=0}^N (\mathbf{Z}_{j(N+1)+i+1, q(N+1)+p+1}) \times \\ & (u_{p,q}^n + u_{p,q}^{n-1}) + r_{15} \sum_{n=1}^{k-1} (\alpha_{k-n-1}^{(2\alpha)} - \alpha_{k-n}^{(2\alpha)}) \times \\ & u_{i,j}^{n-1} \sum_{q=0}^N \sum_{p=0}^N (\mathbf{P}_{j(N+1)+i+1, q(N+1)+p+1}) (u_{p,q}^n + u_{p,q}^{n-1}) \end{aligned} \quad (25)$$

式中 \mathbf{E} 为 $(N+1) \times (N+1)$ 维的单位矩阵。此外

$$\begin{aligned} & (\mathbf{G}_1)_{j(N+1)+i+1, q(N+1)+p+1} = \left(\frac{1}{2} + r_{13} + r_{15} \right) \times \\ & u_{i,j}^{k-1} \mathbf{Z}_{j(N+1)+i+1, q(N+1)+p+1} + \left(\frac{1}{2} + r_{13} + r_{15} \right) \times \\ & v_{i,j}^{k-1} \mathbf{P}_{j(N+1)+i+1, q(N+1)+p+1} - \left(\frac{1}{2\text{Re}} + r_{11} \right) \times \\ & \mathbf{Q}_{j(N+1)+i+1, q(N+1)+p+1} + \left(\frac{1}{\Delta t} + r_{12} + r_{14} \right) \times \\ & \mathbf{E}_{j(N+1)+i+1, q(N+1)+p+1} \end{aligned} \quad (26)$$

当 $0.5 < \alpha < 1, 0 < \beta < 1$, 即 $1 < 2\alpha < 2, 0 < \beta < 1$ 时, 方程(6)的离散格式为

$$\begin{aligned} & \sum_{q=0}^N \sum_{p=0}^N (\mathbf{G}_2)_{j(N+1)+i+1, q(N+1)+p+1} u_{p,q}^k = \\ & r_{16} x_i [(k-1)t]^{c-1} + s^2 x_i [(k-1)t]^{2c} + \\ & r_{17} \lambda_1^\alpha x_i [(k-1)t]^{c-\alpha-1} + r_{18} \lambda_1^\alpha x_i [(k-1)t]^{2c-\alpha} + \\ & r_{17} \lambda_2^\alpha x_i [(k-1)t]^{c-2\alpha-1} + r_{18} \lambda_2^\alpha x_i [(k-1)t]^{2c-2\alpha} + \\ & r_{21} (B_1)_{i,j}^k + r_{21} (B_2)_{i,j}^k - \left(\frac{1}{2} + r_{13} - r_{22} \right) \times \\ & u_{i,j}^{k-1} \sum_{q=0}^N \sum_{p=0}^N (\mathbf{Z}_{j(N+1)+i+1, q(N+1)+p+1}) u_{p,q}^{k-1} - \\ & \left(\frac{1}{2} + r_{13} - r_{22} \right) u_{i,j}^{k-1} \times \\ & \sum_{q=0}^N \sum_{p=0}^N (\mathbf{P}_{j(N+1)+i+1, q(N+1)+p+1}) u_{p,q}^{k-1} + \\ & \left(\frac{1}{2\text{Re}} + r_{11} \right) \sum_{q=0}^N \sum_{p=0}^N (\mathbf{Q}_{j(N+1)+i+1, q(N+1)+p+1}) u_{p,q}^{k-1} + \\ & \left(\frac{1}{\Delta t} + r_{12} + r_{21} - r_{21} \right) \times \\ & \sum_{q=0}^N \sum_{p=0}^N (\mathbf{E}_{j(N+1)+i+1, q(N+1)+p+1}) u_{p,q}^{k-1} - \\ & r_{11} \alpha_{k-1}^{(\beta)} \sum_{q=0}^N \sum_{p=0}^N (\mathbf{Q}_{j(N+1)+i+1, q(N+1)+p+1}) \times \\ & (u_{p,q}^1 + u_{p,q}^0) - r_{11} \sum_{q=0}^N \sum_{p=0}^N (\mathbf{Q}_{j(N+1)+i+1, q(N+1)+p+1}) \times \\ & \sum_{n=1}^{k-1} (\alpha_{k-n-1}^{(\beta)} - \alpha_{k-n}^{(\beta)}) (u_{p,q}^n + u_{p,q}^{n-1}) + \\ & r_{13} \sum_{n=1}^{k-1} (\alpha_{k-n-1}^{(\alpha)} - \alpha_{k-n}^{(\alpha)}) u_{i,j}^{n-1} \times \\ & \sum_{q=0}^N \sum_{p=0}^N (\mathbf{Z}_{j(N+1)+i+1, q(N+1)+p+1}) (u_{p,q}^n + u_{p,q}^{n-1}) + \\ & r_{13} \sum_{n=1}^{k-1} (\alpha_{k-n-1}^{(\alpha)} - \alpha_{k-n}^{(\alpha)}) u_{i,j}^{n-1} \times \\ & \sum_{q=0}^N \sum_{p=0}^N (\mathbf{P}_{j(N+1)+i+1, q(N+1)+p+1}) (u_{p,q}^n + u_{p,q}^{n-1}) + \\ & r_{22} \sum_{n=1}^{k-1} (b_{k-n-1}^{(2\alpha)} - b_{k-n}^{(2\alpha)}) u_{i,j}^{n-1} \times \\ & \sum_{q=0}^N \sum_{p=0}^N (\mathbf{Z}_{j(N+1)+i+1, q(N+1)+p+1}) (u_{p,q}^n - u_{p,q}^{n-1}) + \\ & r_{22} \sum_{n=1}^{k-1} (b_{k-n-1}^{(2\alpha)} - b_{k-n}^{(2\alpha)}) u_{i,j}^{n-1} \times \\ & \sum_{q=0}^N \sum_{p=0}^N (\mathbf{P}_{j(N+1)+i+1, q(N+1)+p+1}) (u_{p,q}^n - u_{p,q}^{n-1}) \end{aligned} \quad (27)$$

式中 E 为 $(N+1) \times (N+1)$ 维的单位矩阵。此外

$$\begin{aligned} (\mathbf{G}_2)_{j(N+1)+i+1,q(N+1)+p+1} &= \left(\frac{1}{2} + r_{13} + r_{22} \right) \times \\ &u_{i,j}^{k-1} \mathbf{Z}_{j(N+1)+i+1,q(N+1)+p+1} + \\ &\left(\frac{1}{2} + r_{13} + r_{22} \right) v_{i,j}^{k-1} \mathbf{P}_{j(N+1)+i+1,q(N+1)+p+1} - \\ &\left(\frac{1}{2\text{Re}} + r_{11} \right) \mathbf{Q}_{j(N+1)+i+1,q(N+1)+p+1} + \\ &\left(\frac{1}{\Delta t} + r_{12} + r_{21} \right) \mathbf{E}_{j(N+1)+i+1,q(N+1)+p+1} \end{aligned} \quad (28)$$

而且

$$\begin{aligned} r_{11} &= \frac{\Delta t^{-\beta} \lambda_3^\beta}{2\text{Re}\Gamma(2-\beta)}, r_{12} = \frac{\Delta t^{-\alpha-1} \lambda_1^\alpha}{\Gamma(2-\alpha)}, r_{13} = \frac{\Delta t^{-\alpha} \lambda_1^\alpha}{2\Gamma(2-\alpha)} \\ r_{14} &= \frac{\Delta t^{-2\alpha-1} \lambda_2^\alpha}{\Gamma(2-2\alpha)}, r_{15} = \frac{\Delta t^{-2\alpha} \lambda_2^\alpha}{2\Gamma(2-2\alpha)}, r_{16} = \frac{sc}{Fu} \\ r_{17} &= \frac{sc\Gamma(c)}{Fu\Gamma(c-\alpha)}, r_{18} = \frac{s^2\Gamma(2c+1)}{\Gamma(2c+1-\alpha)} \\ r_{21} &= \frac{\Delta t^{-2\alpha-1} \lambda_2^\alpha}{\Gamma(3-2\alpha)}, r_{22} = \frac{\Delta t^{-2\alpha} \lambda_2^\alpha}{\Gamma(3-2\alpha)} \\ (A_1)_{i,j}^k &= \sum_{n=1}^{k-1} (\alpha_{k-n-1}^{(\alpha)} - \alpha_{k-n}^{(\alpha)}) (u_{i,j}^n - u_{i,j}^{n-1}) + \\ &\alpha_{k-1}^{(\alpha)} (u_{i,j}^1 - u_{i,j}^0) \\ (A_2)_{i,j}^k &= \sum_{n=1}^{k-1} (\alpha_{k-n-1}^{(2\alpha)} - \alpha_{k-n}^{(2\alpha)}) (u_{i,j}^n - u_{i,j}^{n-1}) + \\ &\alpha_{k-1}^{(2\alpha)} (u_{i,j}^1 - u_{i,j}^0) \\ (B_1)_{i,j}^k &= \sum_{n=1}^{k-1} (b_{k-n-1}^{(2\alpha)} - b_{k-n}^{(2\alpha)}) (u_{i,j}^n - u_{i,j}^{n-1}) \\ (B_2)_{i,j}^k &= \sum_{n=1}^{k-2} (b_{k-n-2}^{(2\alpha)} - b_{k-n-1}^{(2\alpha)}) (u_{i,j}^n - u_{i,j}^{n-1}) \end{aligned}$$

初始条件和边界条件(12~14)的离散格式与上述类似。最后,得到了控制方程组(10~14)的离散格式。

4 收敛性

通过构造数值算例验证离散格式的收敛性^[23,24]。当 $\alpha = 0.1, \beta = 0.1, \lambda_1 = 0.2, \lambda_2 = 0.1, \lambda_3 = 0.3, \text{Re} = 6, Fu = 0.6, s = 0.08, c = 2.2$ 时, 模型表示为

$$\partial u / \partial x + \partial v / \partial y = 0 \quad (29)$$

$$\begin{aligned} (1 + \lambda_1^\alpha D_t^\alpha + \lambda_2^\alpha D_t^{2\alpha}) \left(\frac{1}{Fu} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) &= \\ (1 + \lambda_1^\alpha D_t^\alpha + \lambda_2^\alpha D_t^{2\alpha}) \left(\frac{1}{Fu} \frac{\partial u_\infty}{\partial t} + u_\infty \frac{\partial u_\infty}{\partial x} \right) &+ \\ \frac{1}{\text{Re}} (1 + \lambda_3^\beta D_t^\beta) \frac{\partial^2 u}{\partial y^2} + f(x, y, t) & \end{aligned} \quad (30)$$

$$\begin{aligned} u(x, y, 0) &= v(x, y, 0) = 0, u(0, y, t) = 0 \\ u(0.6, y, t) &= 0, u(x, 0, t) = 0, u(x, 3, t) = 0 \\ (0 \leq x \leq 0.6, 0 \leq y \leq 3, t \geq 0) & \end{aligned} \quad (31)$$

式中

$$\begin{aligned} f(x, y, t) &= \frac{6}{Fu} t x y (0.6 - x) (3 - y) + \\ &18t^4 x y^2 (0.6 - x) (0.3 - x) (3 - y)^2 + \\ &3t^4 x y^2 (0.3 - x) (0.6 - x) (3 - 2y) (2y - 9) + \\ &\frac{432\lambda_1^\alpha}{\Gamma(5-\alpha)} t^{4-\alpha} x y^2 (0.6 - x) (0.3 - x) (3 - y)^2 + \\ &\frac{432\lambda_2^\alpha}{\Gamma(5-2\alpha)} t^{4-2\alpha} x y^2 (0.6 - x) (0.3 - x) (3 - y)^2 + \\ &\frac{6\lambda_1^\alpha}{Fu\Gamma(2-\alpha)} t^{1-\alpha} x y (0.6 - x) (3 - y) - \\ &\frac{12\lambda_3^\beta t^{2-\beta}}{\text{Re}\Gamma(3-\beta)} x (x - 0.6) + \frac{72\lambda_1^\alpha}{\Gamma(5-\alpha)} t^{4-\alpha} x y^2 \times \\ &(0.3 - x) (0.6 - x) (3 - 2y) (2y - 9) + \\ &\frac{72\lambda_2^\alpha}{\Gamma(5-2\alpha)} t^{4-2\alpha} x y^2 (0.3 - x) (0.6 - x) \times \\ &(3 - 2y) (2y - 9) - \frac{6}{\text{Re}} x t^2 (x - 0.6) + \\ &\frac{6\lambda_2^\alpha}{Fu\Gamma(2-2\alpha)} t^{1-2\alpha} x y (0.6 - x) (3 - y) - \\ &\frac{sc}{Fu} x t^{c-1} - s^2 x t^{2c} - \frac{sc\Gamma(c)\lambda_1^\alpha}{Fu\Gamma(c-\alpha)} x t^{c-1-\alpha} - \\ &\frac{s^2\Gamma(2c+1)\lambda_1^\alpha}{\Gamma(2c+1-\alpha)} x t^{2c-\alpha} - \frac{sc\Gamma(c)\lambda_2^\alpha}{Fu\Gamma(c-2\alpha)} x t^{c-1-2\alpha} - \\ &\frac{s^2\Gamma(2c+1)\lambda_2^\alpha}{\Gamma(2c+1-2\alpha)} x t^{2c-2\alpha} \end{aligned} \quad (32)$$

模型(29~31)的解析解为

$$\begin{aligned} \bar{u} &= 3t^2 x y (0.6 - x) (3 - y) \\ \bar{v} &= t^2 y^2 (0.3 - x) (2y - 9) \end{aligned} \quad (33)$$

类似地,式(30)的离散格式为

$$\begin{aligned} \sum_{q=0}^N \sum_{p=0}^N (\mathbf{G}_1)_{j(N+1)+i+1,q(N+1)+p+1} u_{p,q}^k &= \\ r_{16} x_i [(k-1)t]^{c-1} + s^2 x_i [(k-1)t]^{2c} &+ \\ r_{17} \lambda_1^\alpha x_i [(k-1)t]^{c-\alpha-1} + r_{18} \lambda_1^\alpha x_i [(k-1)t]^{2c-\alpha} &+ \\ r_{12} (A_1)_{i,j}^k + r_{14} (A_2)_{i,j}^k + r_{17} \lambda_2^\alpha x_i [(k-1)t]^{c-2\alpha-1} &+ \\ r_{18} \lambda_2^\alpha x_i [(k-1)t]^{2c-2\alpha} - \left(\frac{1}{2} + r_{13} + r_{15} \right) u_{i,j}^{k-1} &\times \\ \sum_{q=0}^N \sum_{p=0}^N (\mathbf{Z}_{j(N+1)+i+1,q(N+1)+p+1}) u_{p,q}^k - \left(\frac{1}{2} + \right. & \\ \left. r_{13} + r_{15} \right) u_{i,j}^{k-1} \sum_{q=0}^N \sum_{p=0}^N (\mathbf{P}_{j(N+1)+i+1,q(N+1)+p+1}) u_{p,q}^k &+ \\ \left(\frac{1}{2\text{Re}} + r_{11} \right) \sum_{q=0}^N \sum_{p=0}^N (\mathbf{Q}_{j(N+1)+i+1,q(N+1)+p+1}) u_{p,q}^k &+ \\ \left(\frac{1}{\Delta t} + r_{12} + r_{14} \right) \sum_{q=0}^N \sum_{p=0}^N (\mathbf{E}_{j(N+1)+i+1,q(N+1)+p+1}) &\times \\ u_{p,q}^{k-1} - r_{11} \sum_{q=0}^N \sum_{p=0}^N (\mathbf{Q}_{j(N+1)+i+1,q(N+1)+p+1}) &\times \\ \sum_{n=1}^{k-1} (\alpha_{k-n-1}^{(\beta)} - \alpha_{k-n}^{(\beta)}) (u_{p,q}^n + u_{p,q}^{n-1}) &- \end{aligned}$$

$$\begin{aligned}
& r_{11} \alpha_{k-1}^{(\beta)} \sum_{q=0}^N \sum_{p=0}^N (\mathbf{Q}_{j(N+1)+i+1, q(N+1)+p+1}) \times \\
& (u_{p,q}^1 + u_{p,q}^0) + r_{13} \sum_{n=1}^{k-1} (\alpha_{k-n-1}^{(\alpha)} - \alpha_{k-n}^{(\alpha)}) u_{i,j}^{n-1} \times \\
& \sum_{q=0}^N \sum_{p=0}^N (\mathbf{Z}_{j(N+1)+i+1, q(N+1)+p+1}) (u_{p,q}^n + u_{p,q}^{n-1}) + \\
& r_{13} \sum_{n=1}^{k-1} (\alpha_{k-n-1}^{(\alpha)} - \alpha_{k-n}^{(\alpha)}) u_{i,j}^{n-1} \times \\
& \sum_{q=0}^N \sum_{p=0}^N (\mathbf{P}_{j(N+1)+i+1, q(N+1)+p+1}) (u_{p,q}^n + u_{p,q}^{n-1}) + \\
& r_{15} \sum_{n=1}^{k-1} (\alpha_{k-n-1}^{(2\alpha)} - \alpha_{k-n}^{(2\alpha)}) u_{i,j}^{n-1} \times \\
& \sum_{q=0}^N \sum_{p=0}^N (\mathbf{Z}_{j(N+1)+i+1, q(N+1)+p+1}) (u_{p,q}^n + u_{p,q}^{n-1}) + \\
& r_{15} \sum_{n=1}^{k-1} (\alpha_{k-n-1}^{(2\alpha)} - \alpha_{k-n}^{(2\alpha)}) u_{i,j}^{n-1} \times \\
& \sum_{q=0}^N \sum_{p=0}^N (\mathbf{P}_{j(N+1)+i+1, q(N+1)+p+1}) \times \\
& (u_{p,q}^n + u_{p,q}^{n-1}) + f(x_i, y_j, t_k) \quad (34)
\end{aligned}$$

$$\begin{aligned}
& (\mathbf{G}_1)_{j(N+1)+i+1, q(N+1)+p+1} = \left(\frac{1}{2} + r_{13} + r_{15} \right) \times \\
& u_{i,j}^{k-1} \mathbf{Z}_{j(N+1)+i+1, q(N+1)+p+1} + \\
& \left(\frac{1}{2} + r_{13} + r_{15} \right) v_{i,j}^{k-1} \mathbf{P}_{j(N+1)+i+1, q(N+1)+p+1} - \\
& \left(\frac{1}{2\text{Re}} + r_{11} \right) \mathbf{Q}_{j(N+1)+i+1, q(N+1)+p+1} + \\
& \left(\frac{1}{\Delta t} + r_{12} + r_{14} \right) \mathbf{E}_{j(N+1)+i+1, q(N+1)+p+1}
\end{aligned}$$

数值解与解析解的最大误差为 $E(\Delta x, \Delta y, \Delta t) = \max_{0 \leq j \leq N} |\bar{u}(x_i, y_j, t_k) - u_{i,j}^k|$, 由表 1 和表 2 可知, 当 $x = x_i, t = t_k$

$t = 0.02, x = 0.6$, 数值解与解析解的最大误差分别随着时间步长和空间步长的减小而减小, 可以说明离散格式是收敛的。

表 1 最大误差随时间步长的变化

Tab. 1 Variation of maximum error with time steps

	Δt	最大误差
$\Delta x = 0.05$	0.01	1.7×10^{-4}
$\Delta y = 0.25$	0.005	8.2×10^{-5}
	0.0025	4.3×10^{-5}

表 2 最大误差随空间步长的变化

Tab. 2 Variation of maximum error with space steps

	Δx	Δy	最大误差
$\Delta t = 0.0025$	0.05	0.25	4.3×10^{-5}
	0.025	0.125	1.4×10^{-5}
	0.0125	0.0625	7.0×10^{-6}

5 结果分析

本节主要讨论物理参数对速度分布的影响, 并通过图像形式给出。 $t = 0.2$ 时速度的三维空间分布如图 2 所示。在边界层内, 越靠近平板, 流体速度越大, 在边界层外的流体速度几乎保持不变。

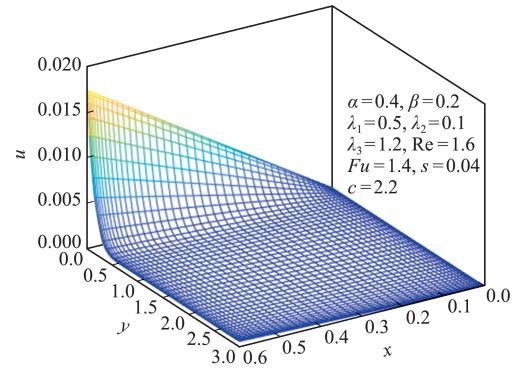


图 2 速度的三维空间分布
Fig. 2 3D spatial distributions of velocity

$x = 0.6, t = 0.2$ 时流体速度随相关参数变化如图 3~图 8 所示。分数阶导数 α 对流体速度的影响如图 3 所示。 α 在分数阶 Burgers 流体模型中, 与表征流体弹性的松弛项有关, α 表示流体相向弹性固相的转变。值得注意的是, 在靠近平板处, 速度场随着 α 的增加而减小, 而远离平板的流体速度出现了相反的趋势。从物理上讲, 说明分数阶方程显示对前一时刻的短期记忆, 并试图返回前一状态, 交点表明分数阶导数的记忆特性。分数阶导数 β 对流体速度的影响如图 4 所示, 增大 β 可以增大流体速度。参数 β 与表征流体粘性阻尼的延迟项有关, 流体的粘性阻尼倾向于提高板的冲击移动引起的速度。 β 越大, 记忆性越小, 有助于促进流体流动。流体速度随不同雷诺数的变化趋势如图 5 所示, 可以看出, 流体速度随着雷诺数的增大

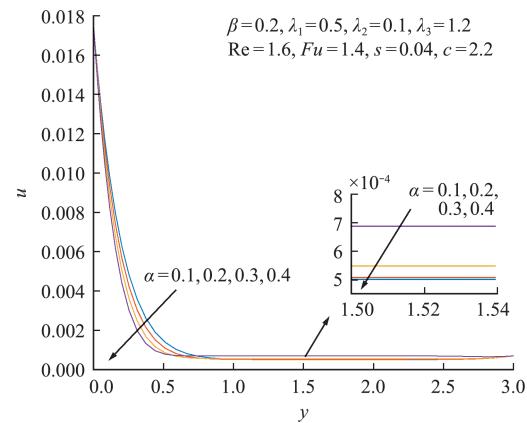


图 3 速度随分数阶导数 α 的变化
Fig. 3 Variety of velocity with fractional derivative α

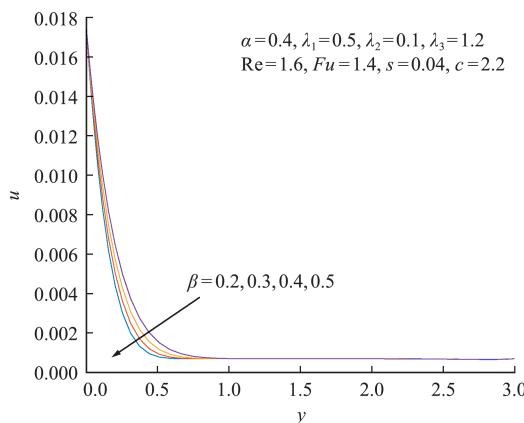


图4 速度随分数阶导数 β 的变化
Fig. 4 Variety of velocity with fractional derivative β

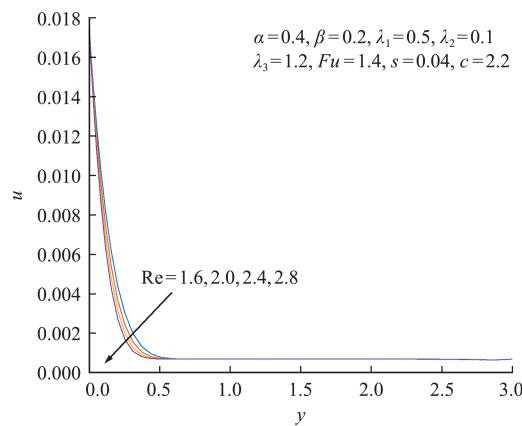


图5 速度随雷诺数 Re 的变化
Fig. 5 Variety of velocity with Reynolds number Re

而减小。雷诺数表示惯性力与粘性力的比值。雷诺数越低,流体的粘度越大,因此流体速度越大。

松弛时间参数 λ_1 对流体速度的影响如图 6 所示,可以看出,在远离平板处,流体速度随着松弛时间参数 λ_1 的增大而增大。然而,在平板附近,随着 λ_1 的增大,增大了流动过程中的速度消耗,导致流体速度减小。从物理角度分析,速度大的流体带动附近流体一起运动,松弛时间 λ_1 越大,流体的粘性越大,粘弹性材料速度快的部分带动速度慢的部分一起运动需要更多的额外时间,从而导致速度的减小。此外,驻点参数 s 对流体速度的影响如图 7 所示。可以看出,不同的驻点参数影响无穷远处的边界条件,从而导致不同的收敛速度。随着驻点参数 s 的增大,速度边界层变厚,流体速度增大。从物理角度分析,驻点参数表示入射流的应变速率与拉伸平板应变速率的比值,驻点参数越大,入射流速度越大,且拉伸平板的应变速率越小,板的拉伸速度对同一位置处流体质点的影响相对减小,入射流快速带动周围流体流动,从而流速加快,边界层变厚。速度受幂指数参数 c 影响的变化趋势如图 8 所示。

幂指数参数 c 决定了平板的速度和来流速度的大小,根据指数函数的单调性,幂指数参数 c 越大,平板的速度和来流速度都增大。

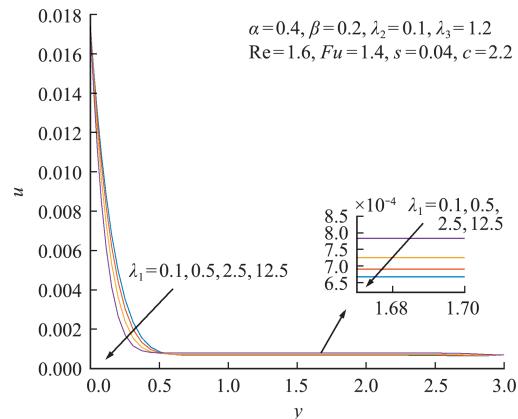


图6 速度随松弛时间参数 λ_1 的变化
Fig. 6 Variety of velocity with relaxation time parameter λ_1

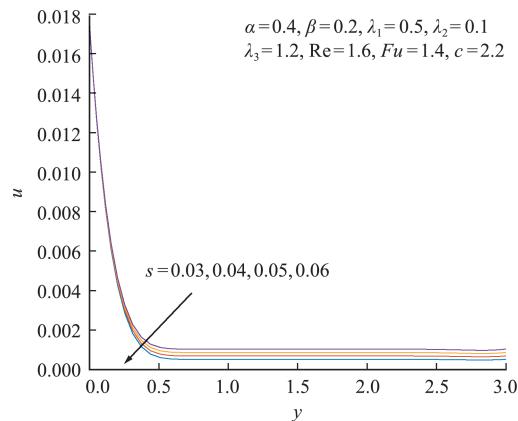


图7 速度随驻点参数 s 的变化
Fig. 7 Variety of velocity with stagnation parameter s

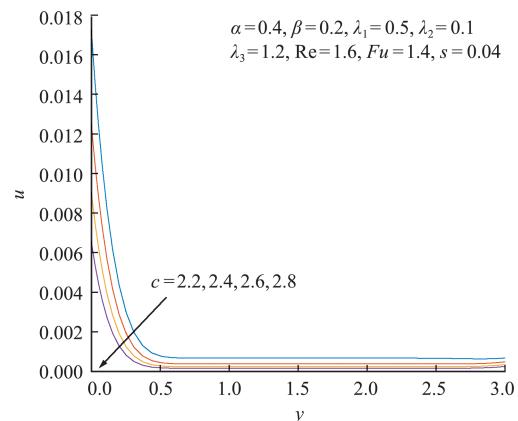


图8 速度随幂指数参数 c 的变化
Fig. 8 Variety of velocity with power exponent parameter c

6 结 论

本文从理论上研究了分数阶 Burgers 流体流经运动平板的二维驻点流动问题。建立了分数阶 Burgers 流体的边界层动量控制方程,由于分数阶

导数参数的取值范围,借助 L-1 算法和 L-2 算法给出了控制方程的两种离散格式,并首次应用有限差分-谱方法求解分数阶 Burgers 流体模型。最后,结合图像分析了相关参数对流体速度的影响。值得注意的是,由于分数阶导数算子的记忆特性, α 取值不同时速度分布出现交点。此外,雷诺数的减小使流体的粘度增大,从而导致流体速度增大。比较松弛时间参数 λ_1 取值不同时的分数阶模型可知,靠近平板处松弛时间参数抑制流体流动,远离平板处松弛时间对流速有促进作用。

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Unsteady stagnation point flow of fractional burgers fluid based on finite difference-spectral method

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Abstract: Unsteady stagnation-point flow of fractional Burgers fluid towards a stretched plate was researched in this paper. Burgers fluid model introduced the fractional derivative, which could better simulate the flow process but increase complexity and solving difficulty of the model. Finite difference-spectral method solved the fractional Burgers fluid model for the first time in this paper, the discrete scheme constructed was simple and effective. Spectral method discretized space terms of the governing equations, finite difference combined with L-1 algorithm and L-2 algorithm separately discretized time terms of the governing equations, and two discrete schemes were given. Results show that velocity decreases near the plate whereas the opposite tendency appears far from plate with increment of the fractional derivative owing to the memory characteristic. In addition, a smaller Reynolds number intensifies fluid viscosity, which magnifies fluid velocity. The relaxation time parameter near the flat plate inhibits the velocity distribution and the relaxation time away from the plate promotes the fluid flow due to the relaxation characteristic.

Key words: unsteady; stagnation-point flow; fractional Burgers fluid; L-2 algorithm; finite difference-spectral method