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基于等效系统假定的变厚度板弯曲广义积分变换解

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摘 要:针对等厚度薄板的弯曲问题,研究人员已给出了基于不同数值算法的经典数值解。针对变厚度薄板弯曲问题的解答较少,且以有限元数值模拟计算为主,计算耗时较大。本文基于广义积分变换原理建立了求解变厚度等效系统的广义积分变换算法,分析了线性和二次变化的变厚度板在多种边界条件下的弯曲问题,利用文献已发表结果同本文建立的广义积分变换解进行验证。计算结果表明,本文建立的基于广义积分变换的变厚度板弯曲求解方法具有较高准确性。同时,通过参数化分析手段,分别利用广义积分变换方法和有限元数值模拟方法讨论了不同边界约束和长宽比等条件对中心点处挠度的影响,计算结果具有较好的一致性,证明本文建立的广义积分变换方法可用于求解变厚度板弯曲问题,且具有较高的准确性。

关键词:变厚度板;广义积分变换;均布载荷;线性载荷

中图分类号:O242;O343 **文献标志码**:A **文章编号**:1007-4708(2022)04-0498-08

1 引 言

在实际工程中,除了等厚度板之外,变厚度板已经逐渐受到工程设计人员的重视。在航空航天器、新型船舰及建筑等设计中,变厚度板已经成为一种重要的结构元件。变厚度板相比于等厚度板有助于减轻结构元件的重量,提高材料的利用率。因此,变厚度板力学性质的研究有着重要的意义,变厚度板力学响应问题也引起了国内外学者的关注。

目前,有限元法、有限差分法[1,2]、积分变换法[3,4]和等几何法[5]等算法广泛应用于求解各向同性板和正交各向同性板弯曲问题。李锐等[6]通过辛-叠加方法求解了地基上矩形中厚板的弯曲问题。Heydari等[7]利用变分法优化了厚度呈线性和二次变化的Pasternak双参数弹性地基上圆形功能梯度板的稳定性方程,并利用谱里兹法求解该稳定性方程,进一步讨论了线性、二次厚度变化系数和材料泊松比等对该圆形功能梯度板屈曲的影响。Zenkour[8,9]采用小参数法讨论了机械外力和湿热载荷作用下Levy型变厚度薄板的弯曲问题。Zhang等[10]利用二维广义有限积分变换方法求解了正交各向异性矩形薄板的弯曲解析解。Li等[11]利用有限积分变换法求解了完全夹紧的正交各向异性矩形薄板在任意载荷下的精确弯曲解。Tian等[12]提出了双有限积分变换法来求解弹性地基上各边自由的中厚矩形板的弯曲解析解。

广义积分变换(GITT)是近年发展起来的一种混合数值分析方法[13],目前该计算方法已广泛应用于求解流动与传热问题[14-16]、固体力学问题[17-20]以及结构振动等问题[21-23]。变厚度板相比于等厚度板其弯曲刚度存在明显的非线性行为,利用广义积分变换方法求解该问题具有明显的优势,目前未见相关的文献报道。本文建立了求解基于等效系统假定的变厚度板弯曲的广义积分变换算法。通过求解满足给定边界条件的辅助方程的特征函数和特征值,利用积分变换方法,将变厚度薄板弯曲问题控制方程变换成常微分方程组并进行求解。将本文计算结果与已发表的结果和有限元数值模拟结果对比,发现本文建立的基于广义积分变换(GITT)的计算结果满足精度要求。利用该算法分析了不同边界条件和载荷分布条件下变厚度薄板的弯曲问题,分析了不同长宽比、边界约束与载荷条件对变厚度板挠度和弯矩的影响规律。

2 变厚度薄板弯曲控制方程

考虑长为 a,宽为 b 的各向同性变厚度矩形薄板,如图1所示。基于小变形假设,横向载荷作用

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下的变厚度板弯曲控制方程为

$$\nabla^2(D\nabla^2 w)-(1-\nu)\left(\frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}-2\frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right)=q(x,y) \tag{1}$$

当考虑板厚度 t 仅沿 y 方向发生变化，即 $t=t(y)$，则方程(1)可简化为

$$D\nabla^4 w=q(x,y)-2\frac{\partial D}{\partial y}\frac{\partial}{\partial y}(\nabla^2 w)-\frac{\partial^2 D}{\partial y^2}\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right) \tag{2}$$

式中　$D=Et^3/12(1-\nu^2)$ 为板的弯曲刚度，ν 为泊松比，w 为挠度，t 为厚度，E 为弹性模量。

若变厚度板的厚度 t 在 y 方向上满足

$$t=t_0[1+kf_N(y)] \tag{3}$$

式中　t_0 为 $y=0$ 时板的厚度，厚度参数 $f_N(y)=(2y/b)^N$，当 $N=1$ 和 2 时，分别表示板厚度沿 y 轴方向呈线性和二次变化，k 为变厚度系数，如图 1 所示。弯曲刚度满足 $D=D_0[1+kf_N(y)]^3$，其中，$D_0=Et_0^3/12(1-\nu^2)$表示厚度为 t_0 的板的抗弯刚度。

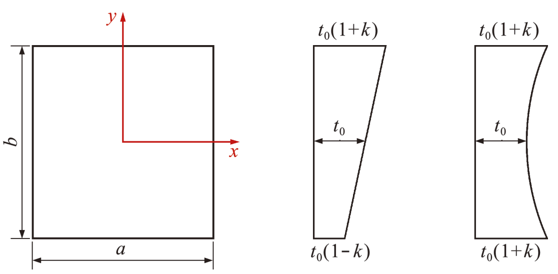

图 1　变厚度板

Fig. 1　Variable thickness plate

变厚度板弯矩方程为

$$M_x=-\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)\times D_0[1+kf_N(y)]^3 \tag{4a}$$

$$M_y=-\left(\nu\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\times D_0[1+kf_N(y)]^3 \tag{4b}$$

为了简化变厚度弯曲控制方程(2)数值求解的复杂性，Fertis 等[24-26]提出了基于小参数假定的变厚度板等效系统。根据该假定，厚度沿 y 方向呈线性和二次分布的弯曲控制方程可简化为

$$\nabla^4 w=\frac{[1-3kf_1(y)]q}{D_0}+6k\left(k\frac{df_1}{dy}f_1-\frac{df_1}{dy}\right)\frac{\partial}{\partial y}(\nabla^2 w_0) \tag{5a}$$

$$\nabla^4 w=\frac{q}{D_0}[1-3kf_2(y)]-k\left[6\frac{df_2}{dy}\frac{\partial}{\partial y}(\nabla^2 w_0)+3\frac{d^2 f_2}{dy^2}\left(\frac{\partial^2 w_0}{\partial y^2}+\nu\frac{\partial^2 w_0}{\partial x^2}\right)\right] \tag{5b}$$

式中　w_0 为厚度为 t_0 的等厚度板在载荷 q 作用下的挠度，满足控制方程

$$\nabla^4 w_0=q/D_0 \tag{5c}$$

如图 2 所示，考虑三种载荷分布形式，即均布载荷 UN，$q(x,y)=q_0$，沿 x 方向线性分布载荷 TX，$q(x,y)=q_0(2x+a)/2a$ 和沿 y 方向线性分布载荷 TY，$q(x,y)=q_0(2y+b)/2b$。同时，考虑六种边界条件，如图 3 所示，C 为固支边界，S 为简支边界条件。

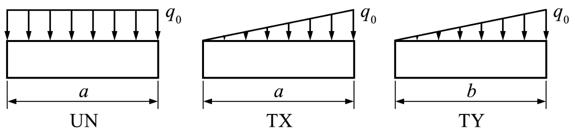

图 2　载荷加载

Fig. 2　Load conditions

引入无量纲量，

$$\tilde{w}=\frac{w}{a^4q_0/D_0},\ \eta=\frac{y}{b/2},\ \xi=\frac{x}{a/2},\ c=\frac{b}{a} \tag{6}$$

方程(5)可表示为无量纲形式，

$$\frac{\partial^4\tilde{w}}{\partial\xi^4}+\frac{2}{c^2}\frac{\partial^4\tilde{w}}{\partial\xi^2\partial\eta^2}+\frac{1}{c^4}\frac{\partial^4\tilde{w}}{\partial\eta^4}=\frac{1-3k\eta}{16}\alpha+6k(k\eta-1)\left(\frac{1}{c^4}\frac{\partial^3\tilde{w}_0}{\partial\eta^2\partial\xi}+\frac{1}{c^2}\frac{\partial^3\tilde{w}_0}{\partial\xi^2\partial\eta}\right) \tag{7a}$$

$$\frac{\partial^4\tilde{w}}{\partial\xi^4}+\frac{2}{c^2}\frac{\partial^4\tilde{w}}{\partial\xi^2\partial\eta^2}+\frac{1}{c^4}\frac{\partial^4\tilde{w}}{\partial\eta^4}=\frac{1-3k\eta^2}{16}\alpha-k\left[12\eta\left(\frac{1}{c^4}\frac{\partial^3\tilde{w}_0}{\partial\eta^3}+\frac{1}{c^2}\frac{\partial^3\tilde{w}_0}{\partial\xi^2\partial\eta}\right)+6\left(\nu\frac{1}{c^2}\frac{\partial^2\tilde{w}_0}{\partial\xi^2}+\frac{1}{c^4}\frac{\partial^2\tilde{w}_0}{\partial\eta^2}\right)\right] \tag{7b}$$

$$\frac{\partial^4\tilde{w}_0}{\partial\xi^4}+\frac{2}{c^2}\frac{\partial^4\tilde{w}_0}{\partial\xi^2\partial\eta^2}+\frac{1}{c^4}\frac{\partial^4\tilde{w}_0}{\partial\eta^4}=\frac{1}{16}\alpha \tag{7c}$$

式中　α 为载荷分布系数，当 α 取 1，$(1+\xi)/2$ 和 $(1+\eta)/2$ 时，方程(7)分别表示变厚度板承受均布载荷 UN、沿 x 轴线性分布载荷 TX 和沿 y 轴线性分布载荷 TY 情况下的控制方程。

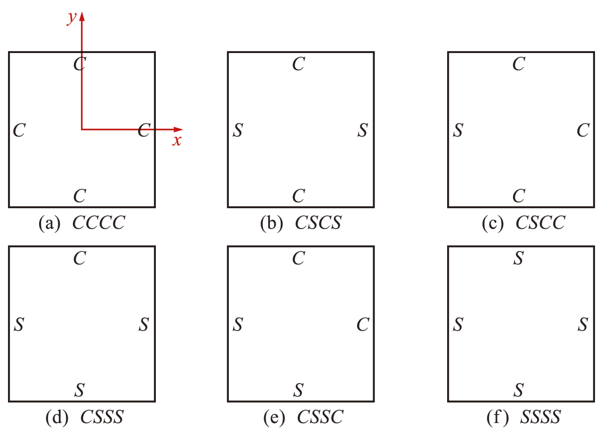

图 3　变厚度板边界条件

Fig. 3　Boundary conditions of plate with variable thickness

同理,弯矩方程的无量纲形式为

$$\widetilde{M}_\xi = -\left(\frac{\partial^2 \widetilde{w}}{\partial \xi^2} + \frac{\nu}{c^2}\frac{\partial^2 \widetilde{w}}{\partial \eta^2}\right)(1 + k\eta^N)^3 \quad (8a)$$

$$\widetilde{M}_\eta = -\left(\nu\frac{\partial^2 \widetilde{w}}{\partial \xi^2} + \frac{1}{c^2}\frac{\partial^2 \widetilde{w}}{\partial \eta^2}\right)(1 + k\eta^N)^3 \quad (8b)$$

式中 当 $N=1$ 和 2 时,分别为厚度沿 y 方向呈线性和二次变化时板内任意点的弯矩。

简支边界条件 S 和固支边界条件 C 满足

$$S: \widetilde{w} = \widetilde{M} = 0, \; C: \widetilde{w} = \widetilde{w}' = 0 \quad (9)$$

3 广义积分变换解的建立

根据广义积分变换的基本原理,选取 η 方向的辅助特征方程

$$\frac{d^4 Y_i(\eta)}{d\eta^4} = \mu_i^4 Y_i(\eta) \quad (-1<\eta<1) \quad (10)$$

不同工况下 η 方向的边界条件以及方程(10)的特征函数 $Y_i(\eta)$ 和特征值 μ_i 可通过方程(11~16)求得。

(1) 两端固定边界条件 CC

$$Y_i(-1) = Y_i'(-1) = Y_i(1) = Y_i'(1) = 0 \quad (11)$$

方程(10)满足上述边界条件的特征函数 $Y_i(\eta)$,见式(12a),特征值 μ_i 由式(12b)求得,

$$Y_i(\eta) = \begin{cases} \dfrac{\cos(\mu_i\eta)}{\cos(\mu_i)} - \dfrac{\cosh(\mu_i\eta)}{\cosh(\mu_i)} & (i=1,3,5,\cdots) \\ \dfrac{\sin(\mu_i\eta)}{\sin(\mu_i)} - \dfrac{\sinh(\mu_i\eta)}{\sinh(\mu_i)} & (i=2,4,6,\cdots) \end{cases} \quad (12a)$$

$$(-1)^i \tanh(\mu_i) = \tan(\mu_i) \quad (i=1,2,3,\cdots) \quad (12b)$$

(2) 一端简支一端固支边界条件 SC

$$Y_i(-1) = Y_i''(-1) = Y_i(1) = Y_i'(1) = 0 \quad (13)$$

此边界条件下,方程(10)的特征函数 $Y_i(\eta)$,见式(14a),特征值 μ_i 可由式(14b)求得,

$$Y_i(\eta) = \frac{\sin \mu_i(\eta+1)}{\sin(2\mu_i)} - \frac{\sinh \mu_i(\eta+1)}{\sinh(2\mu_i)} \quad (i=1,2,3,\cdots) \quad (14a)$$

$$\tanh(2\mu_i) = \tan(2\mu_i) \quad (i=1,2,3,\cdots) \quad (14b)$$

(3) 两端简支边界条件 SS

$$Y_i(-1) = Y_i''(-1) = Y_i(1) = Y_i''(1) = 0 \quad (15)$$

此边界条件下,方程(10)的特征函数 $Y_i(\eta)$ 和特征值 μ_i 为

$$Y_i(\eta) = \begin{cases} \cos(\mu_i\eta) & (i=1,3,5,\cdots) \\ \sin(\mu_i\eta) & (i=2,4,6,\cdots) \end{cases} \quad (16a)$$

$$\mu_i = i\pi/2 \quad (i=1,2,3,\cdots) \quad (16b)$$

特征向量 $\mathbf{Y}_i$ 满足正交特性,

$$\int_{-1}^{1} \mathbf{Y}_i(\eta) Y_j(\eta) d\eta = \delta_{ij} N_i, \; \delta_{ij} = \begin{cases} 1 & (i=j) \\ 0 & (i \neq j) \end{cases} \quad (17)$$

建立归一化表达式为

$$\widetilde{X}_i(\eta) = \widetilde{Y}_i(\eta) = \mathbf{Y}_i(\eta)/\sqrt{N_i} \quad (i=1,2,3,\cdots) \quad (18)$$

建立方程(7a,7b)的积分变换与逆变换如下,

$$\overline{w}_i(\xi) = \int_{-1}^{1} \widetilde{w}_i(\xi,\eta) \widetilde{X}_i(\eta) d\eta$$

$$\widetilde{w}_i(\xi,\eta) = \sum_{i=1}^{\infty} \widetilde{X}_i(\eta) \overline{w}_i(\xi) \quad (19a)$$

同理,方程(7c)的积分变换与逆变换的表达式为

$$\overline{w}_{0i}(\xi) = \int_{-1}^{1} \widetilde{w}_{0i}(\xi,\eta) \widetilde{Y}_i(\eta) d\eta$$

$$\widetilde{w}_{0i}(\xi,\eta) = \sum_{i=1}^{\infty} \widetilde{Y}_i(\eta) \overline{w}_{0i}(\xi) \quad (19b)$$

将原方程(7)乘以特征函数并在[-1,1]区间积分,利用逆变换公式化简方程,原偏微分方程(7)可转化为常微分方程组形式为

$$\frac{d^4\overline{w}_i}{d\xi^4} + \frac{2}{c^2}\sum_{j=1}^{\infty} A_{ij} \frac{d^2\overline{w}_j}{d\xi^2} + \frac{1}{c^4}\overline{w}_i \mu_i^4 = B_i + \frac{1}{c^4}\sum_{j=1}^{\infty} C_{ij}\overline{w}_{0j} + \frac{1}{c^2}\sum_{j=0}^{\infty} D_{ij}\frac{d^2\overline{w}_{0j}}{d\xi^2} \quad (20a)$$

$$\frac{d^4\overline{w}_i}{d\xi^4} + \frac{2}{c^2}\sum_{j=1}^{\infty} E_{ij} \frac{d^2\overline{w}_j}{d\xi^2} + \frac{\mu_i^4\overline{w}_i}{c^4} = F_i - k\left(\frac{12}{c^4}\sum_{j=1}^{\infty} G_{ij}\overline{w}_{0j} + \frac{12}{c^2}\sum_{j=0}^{\infty} H_{ij}\frac{d^2\overline{w}_{0j}}{d\xi^2} + \frac{6\nu}{c^2}\sum_{j=0}^{\infty} N_{ij}\frac{d^2\overline{w}_{0j}}{d\xi^2} + \frac{6}{c^4}\sum_{j=0}^{\infty} U_{ij}\overline{w}_{0j}\right) \quad (20b)$$

$$\frac{d^4\overline{w}_{0i}}{d\xi^4} + \frac{2}{c^2}\sum_{j=1}^{\infty} V_{ij}\frac{d^2\overline{w}_{0j}}{d\xi^2} + \frac{1}{c^4}\phi_i^4 \overline{w}_{0i} = Z_i \quad (20c)$$

式中

$$A_{ij} = \int_{-1}^{1} \widetilde{X}_i \frac{d^2\widetilde{X}_j}{d\eta^2} d\eta, \; B_i = \int_{-1}^{1} \frac{1-3k\eta}{16} \alpha \widetilde{X}_i d\eta$$

$$C_{ij} = \int_{-1}^{1} 6k(k\eta - 1)\widetilde{X}_i \frac{d^3\widetilde{Y}_j}{d\eta^3} d\eta, \; Z_i = \int_{-1}^{1} \frac{\alpha}{16}\widetilde{Y}_i d\eta$$

$$E_{ij} = \int_{-1}^{1} \widetilde{X}_i \frac{d^2\widetilde{X}_j}{d\eta^2} d\eta, \; V_{ij} = \int_{-1}^{1} \widetilde{Y}_i \frac{d^2\widetilde{Y}_j}{d\eta^2} d\eta$$

$$D_{ij} = \int_{-1}^{1} 6k(k\eta - 1)\widetilde{X}_i \frac{d\widetilde{Y}_j}{d\eta} d\eta, \; N_{ij} = \int_{-1}^{1} \widetilde{Y}_i \widetilde{X}_j d\eta$$

$$F_i = \int_{-1}^{1} \frac{1-3k\eta^2}{16} \alpha \widetilde{X}_i d\eta, \; G_{ij} = \int_{-1}^{1} \eta \widetilde{X}_i \frac{d^3\widetilde{Y}_j}{d\eta^3} d\eta$$

$$H_{ij}=\int_{-1}^{1}\eta\widetilde{X}_i\frac{\mathrm{d}\widetilde{Y}_j}{\mathrm{d}\eta}\mathrm{d}\eta,\ U_{ij}=\int_{-1}^{1}\widetilde{X}_i\frac{d^2\widetilde{Y}_j}{\mathrm{d}\eta^2}\mathrm{d}\eta$$

同理，积分变化后的弯矩方程可表示为

$$M_\xi=-\Big(\sum_{i=1}^{\infty}\frac{d^2\overline{w}_i}{\mathrm{d}\xi^2}\widetilde{Y}_i+\frac{\nu}{c^2}\sum_{i=1}^{\infty}\frac{d^2\widetilde{Y}_i}{\mathrm{d}\eta^2}\overline{w}_i\Big)(1+k\eta^N)^3 \tag{21a}$$

$$M_\eta=-\Big(\nu\sum_{i=1}^{\infty}\frac{d^2\overline{w}_i}{d\xi^2}\widetilde{Y}_i+\frac{1}{c^2}\sum_{k=1}^{\infty}\frac{d^2\widetilde{Y}_i}{d\eta^2}\overline{w}_i\Big)(1+k\eta^N)^3 \tag{21b}$$

式中 当 N＝1 和 2 时，分别为沿 y 方向呈线性和二次变化的变厚度板内任意点弯矩。

考虑 ξ 方向不同边界条件如下。

两端固定边界条件 CC，

$$\overline{w}_i(-1)=\overline{w}_i'(-1)=\overline{w}_i(1)=\overline{w}_i'(1)=0 \tag{22a}$$

一端简支一端固支边界条件 SC，

$$\overline{w}_i(-1)=\overline{w}_i''(-1)=\overline{w}_i(1)=\overline{w}_i'(1)=0 \tag{22b}$$

两端简支边界条件 SS，

$$\overline{w}_i(-1)=\overline{w}_i''(-1)=\overline{w}_i(-1)=\overline{w}_i''(1)=0 \tag{22c}$$

通过设置合适的截断阶数，利用 Mathematica 求得各方程系数，通过 NDSolve 求解截断后的常微分方程组(20)满足边界条件(22)的解 $\overline{w}_i(\xi)$，利用逆变换公式 $\widetilde{w}_i(\xi,\eta)=\sum_{i=1}^{\infty}\widetilde{X}_i(\eta)\overline{w}_i(\xi)$ 可求得无量纲挠度 $\widetilde{w}_i(\xi,\eta)$。同理，利用弯矩计算式(21)，可求得无量纲弯矩 $M_\xi(\xi,\eta)$ 和 $M_\eta(\xi,\eta)$。

表 1　变厚度板挠度 $\widetilde{w}(a^4q_0/100D_0)$、弯矩 $\widetilde{M}_x(q_0a^2/10)$ 和 $\widetilde{M}_y(q_0a^2/10)$ 计算结果及其与文献[8]数据的对比

Tab. 1　Comparison of the deflection and moments with the data from literature [8]

载荷形式	c		N=0			N=1			N=2		
			$\widetilde{w}(0,0)$	$\widetilde{M}_x(0,0)$	$\widetilde{M}_y(0,0)$	$\widetilde{w}(0,0)$	$\widetilde{M}_x(0,0)$	$\widetilde{M}_y(0,0)$	$\widetilde{w}(0,0)$	$\widetilde{M}_x(0,0)$	$\widetilde{M}_y(0,0)$
UN	1.0	本文	0.1917	0.2443	0.3338	0.1962	0.2498	0.3392	0.1486	0.1964	0.2956
		文献[8]	0.1917	0.2439	0.332	0.1944	0.2428	0.3215	0.1548	0.2029	0.2995
	1.5	本文	0.5327	0.5859	0.4633	0.5406	0.5941	0.4672	0.4438	0.4993	0.4360
		文献[8]	0.5326	0.5848	0.4595	0.5393	0.5857	0.4432	0.4538	0.5038	0.4368
	2.0	本文	0.8447	0.8708	0.4809	0.8519	0.8778	0.4825	0.7413	0.7760	0.4711
		文献[8]	0.8445	0.8687	0.4736	0.8539	0.8717	0.4560	0.7506	0.7838	0.4692
	3.0	本文	1.1691	1.1485	0.4364	1.1722	1.1513	0.4370	1.0843	1.0735	0.4380
		文献[8]	1.1681	1.1436	0.4213	1.1782	1.1482	0.4065	1.0896	1.0770	0.4331
TX	1.0	本文	0.0959	0.1221	0.1669	0.0981	0.1249	0.1696	0.1486	0.1964	0.2956
		文献[8]	0.0959	0.1219	0.1662	0.0972	0.1214	0.1607	0.1548	0.2029	0.2995
	1.5	本文	0.2663	0.2929	0.2317	0.2703	0.2970	0.2336	0.4438	0.4993	0.4360
		文献[8]	0.2663	0.2924	0.2297	0.2697	0.2929	0.2216	0.4538	0.5038	0.4368
	2.0	本文	0.4223	0.4353	0.2404	0.4239	0.4388	0.2412	0.7413	0.7760	0.4711
		文献[8]	0.4223	0.4343	0.2368	0.4269	0.4359	0.2280	0.7506	0.7838	0.4692

4　算例与结果分析

表 1 为等厚度板（N＝0）、厚度沿 y 轴线性（N＝1）和二次（N＝2）变化的变厚度板中心点处的挠度和弯矩计算结果。该算例中，边界条件为 y 方向对边固支 x 方向对边简支（CSCS）边界条件，载荷边界条件为均布载荷（UN）和沿 x 方向线性分布载荷 TX，泊松比为 0.3，变厚度系数 $k=0.2$。通过与文献[8]的结果对比，利用 GITT 方法求得的变厚度板中心处的挠度和弯矩与文献的结果吻合较好。

同时，利用 GITT 方法计算了不同长宽比、载荷分布（UN、TX 和 TY 载荷）以及边界条件下的变厚度板弯曲问题。图 4～图 6 分别表示在不同边界条件（CCCC，CSCC，CSCS，CSSS，CSSC，SSSS）和载荷分布条件（UN，TX 及 TY）下变厚度板弯曲挠度的 GITT 计算结果（ANA. N）和有限元数值模拟结果（FEM. N）的比较，其中，N＝1 和 N＝2 分别表示变厚度板的厚度沿 y 轴线性和二次分布。假定泊松比为 0.3，变厚度系数 $k=0.2$。可以看出，基于 GITT 的计算结果（ANA. N）与有限元数值模拟结果（FEM. N）吻合较好。

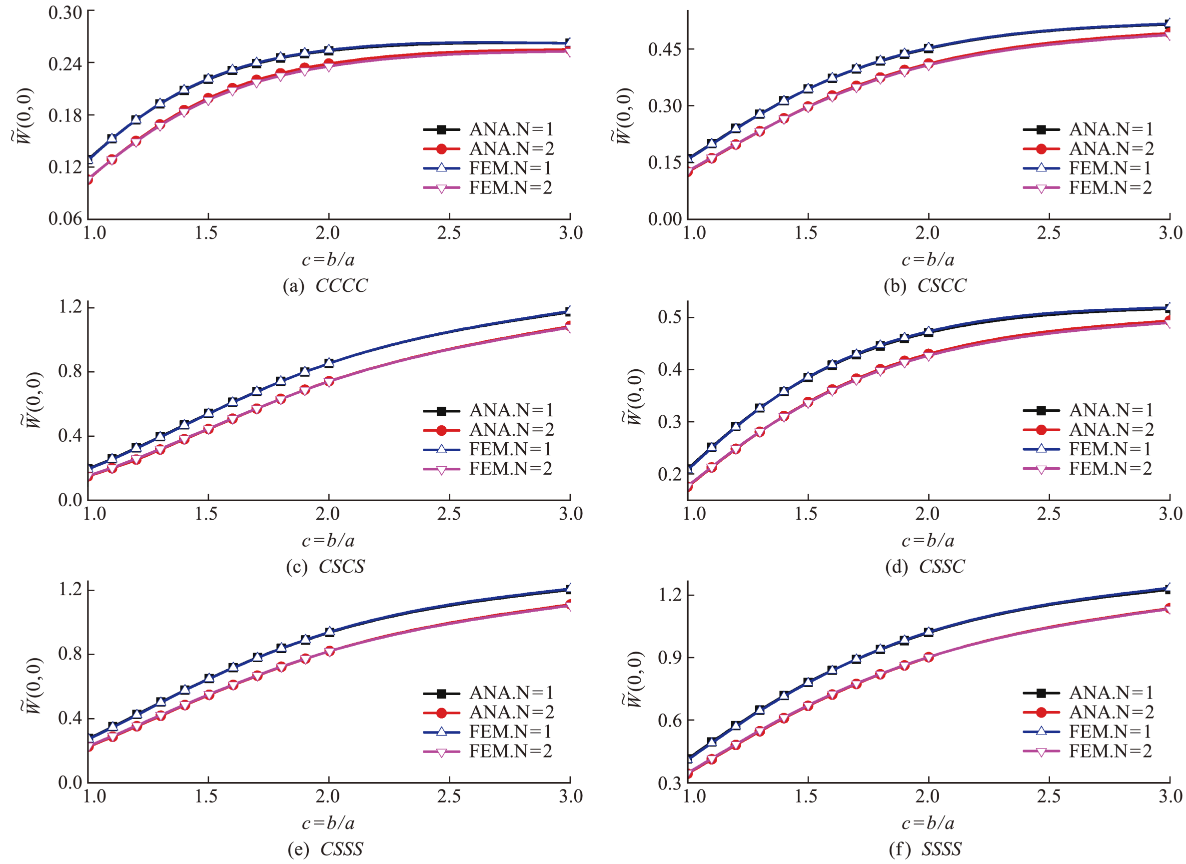

图 4 UN 载荷作用下变厚度板挠度 $\tilde{w}(a^4q_0/100D_0)$ 与有限元数值模拟结果对比

Fig. 4 Comparison of the deflection $\tilde{w}(a^4q_0/100D_0)$ between the GITT and FEM simulation results under UN load

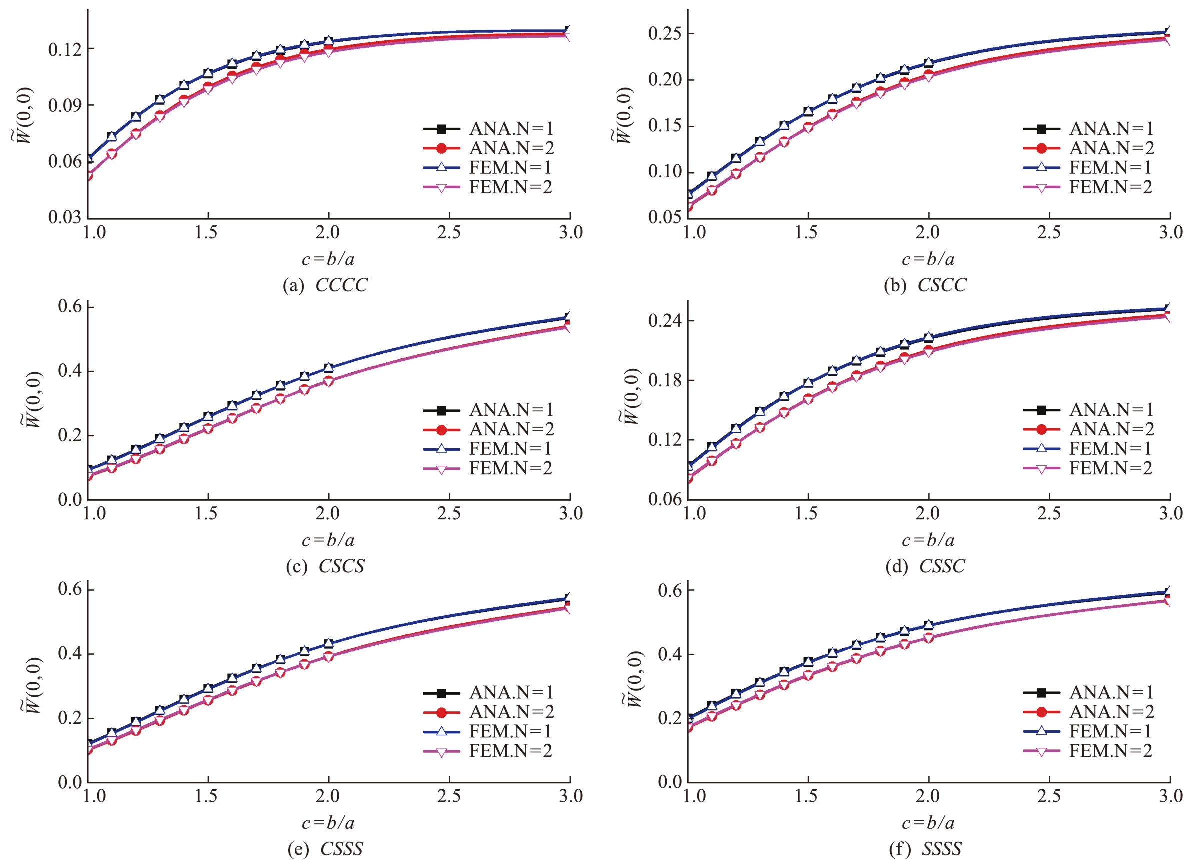

图 5 TY 载荷作用下变厚度板挠度 $\tilde{w}(a^4q_0/100D_0)$ 与有限元数值模拟结果对比

Fig. 5 Comparison of $\tilde{w}(a^4q_0/100D_0)$ between the GITT and FEM simulation results under TX load

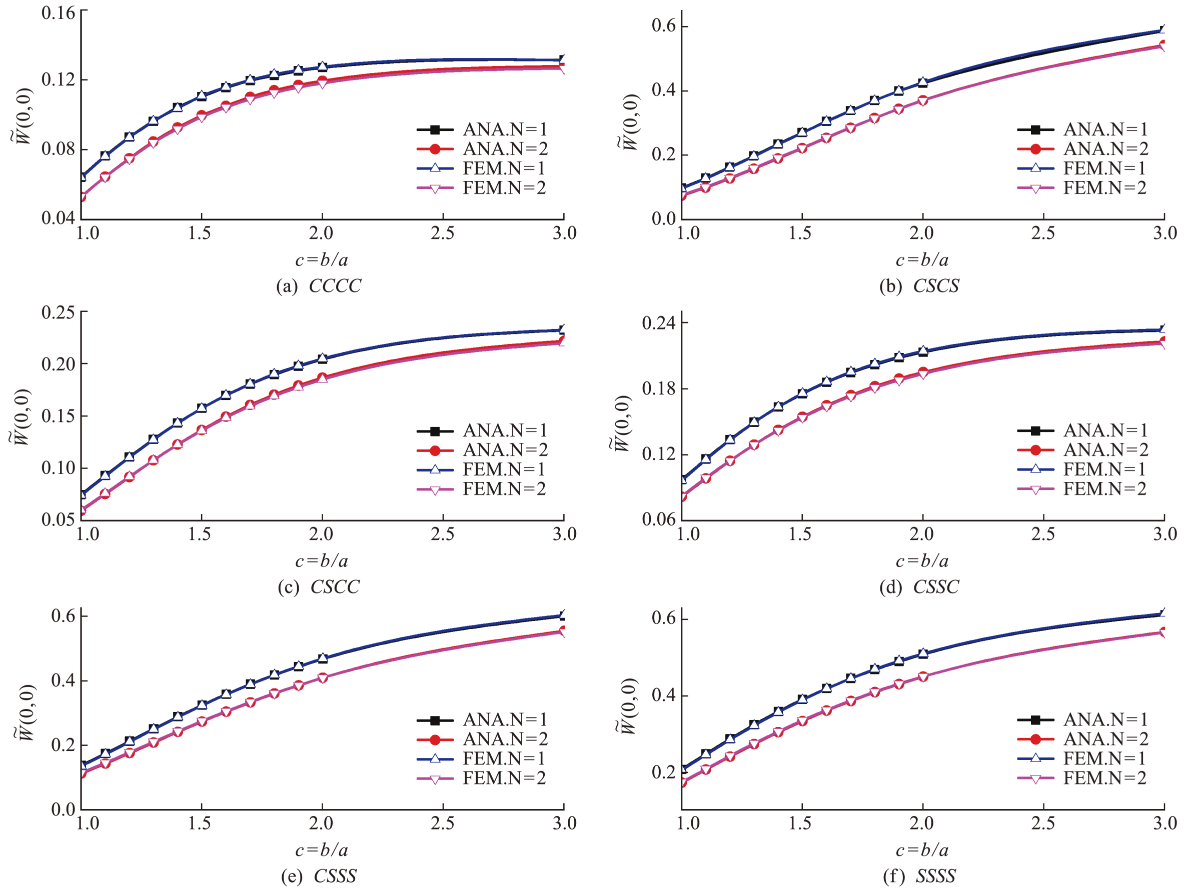

图 6　TX 载荷和多种边界下变厚度板挠度 $\tilde{w}(a^4q_0/100D_0)$ 与有限元模拟结果对比

Fig. 6　Comparison of $\tilde{w}(a^4q_0/100D_0)$ between the GITT and FEM simulation results under TY load

5　结　论

本文基于广义积分变换原理，建立了求解变厚度板弯曲问题的广义积分变换法 GITT 算法。通过与已发表的文献数据和有限元数值模拟计算结果比较，证明建立的 GITT 方法具有较高的计算精度。利用建立的广义积分变换法算法分析了线性和二次变化的变厚度板在均布载荷 UN、载荷沿 x 方向和 y 方向线性分布条件下的挠度和弯矩值，同时讨论了不同边界约束条件和长宽比等几何特征对中心点处挠度的影响规律，计算结果可为实际工程计算提供参考。

本文以厚度沿 y 方向变化为例建立了求解的变厚度板弯曲问题的 GITT 模型，该方法同样适用于求解厚度沿 x 和 y 两方向变化以及任意载荷作用下的变厚度板弯曲问题。

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Bending solution for equivalent system of variable thickness plate by the generalized integral transform technique

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Abstract: For the bending problem of thin plates with uniform thickness, some classical numerical solutions based on various numerical techniques have been developed by researchers. However, the bending problem solutions of plates with variable thickness have been rarely reported, and most of them were based on finite element calculation, which consumes a lot of CPU time. In the present work, the authors developed a hybrid numerical and analytical mothed based on the Generalized Integral Transform Technique (GITT) to solve the bending problem of plates with variable thickness of linear and quadratic variations. The calculated results were calibrated by the results which were reported in the literatures. Finally, the effects of boundary constraints and aspect ratio on the deflection at the center point is discussed by using the GITT method and finite element numerical simulation method, respectively and the results are in good agreement. It can be proven that the generalized integral transformation method established in this paper can be used to solve the bending problem of variable thickness plate with good accuracy.

Key words: variable thickness plate; generalized integral transform technique; uniform load; linear load

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