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四角点支承四边自由矩形薄板屈曲问题的新解析解

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摘 要：角点支承矩形薄板的屈曲问题是板壳力学的一类重要课题，控制方程和边界条件的复杂性导致寻求该类问题的解析解十分困难。虽然各类近似/数值方法可用于解决此类难题，但作为基准的精确解析解在公开文献中鲜有报道。本文基于近年来提出的辛叠加方法，解析求解了四角点支承四边自由矩形薄板的屈曲问题。首先将问题拆分为两个子问题，接着利用分离变量与辛本征展开推导出子问题的解析解，最后通过叠加获得原问题的解。由于求解过程从基本控制方程出发，逐步严格推导，无需假定解的形式，因此本文解法是一种理性的解析方法。数值算例给出了不同长宽比和不同面内载荷比情况下，四角点支承四边自由矩形薄板的屈曲载荷和典型屈曲模态，并经有限元方法验证，确认了解析解的正确性。

关键词：辛叠加方法；矩形薄板；屈曲；角点支承

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1 引 言

角点支承的矩形薄板作为一种常见结构形式，广泛应用于建筑结构、机械构件和航空航天器等工程实际。板的屈曲作为典型的力学失效形式之一，在过去几十年受到了广泛关注，该类问题的解析求解在理论与实际中都具有重要意义。对于板的线性屈曲问题，核心是在给定的边界条件下，通过求解高阶偏微分控制方程，得到对结构设计具有重要参考价值的屈曲载荷和相应的屈曲模态。然而，由于数学上的复杂性，许多问题难以得到同时满足高阶偏微分方程和边界条件的解析解。现有的矩形板的屈曲解析解主要局限于两对边简支的情况，即Lévy型板，而对于角点支承条件下的非对边简支的矩形板，绝大部分研究都是基于有限差分法[1]、微分求积法[2]、离散奇异卷积法[3,4]、无网格法[5]和广义伽辽金法[6]等近似/数值方法，而关于解析方法和解析解的报道较少。

钟万勰院士[7-9]将辛数学思想引入弹性力学中，为弹性力学求解开辟了新思路，由此产生的辛弹性力学方法已在结构皱褶[10]、断裂[11]和弹性波[12]等众多领域中得到了充分的应用。在辛弹性力学的基础上，李锐等[13-17]针对复杂板壳力学问题提出了一种新的解析方法，即辛叠加方法，获得了若干板壳结构弯曲、振动和屈曲问题的新解析解。辛叠加方法的基本思想是将待解决的问题转化为几个可由辛数学法求解的子问题的叠加。该方法的求解过程是在基于辛空间的哈密顿体系中进行，而不是在传统的基于欧几里德空间的拉格朗日体系中进行；解析解是通过直接的严格推导得到，不需要对解的形式做任何假定，这是经典的半逆方法难以做到的；同时，该方法规避了单纯采用辛数学方法出现的本征方程难以解析求解等瓶颈，在解析求解含有复杂边界的板壳振动和屈曲问题中具有独特优势。

本文首次采用辛叠加方法解析求解四角点支承四边自由矩形薄板的屈曲问题——由于边界条件的复杂性，此类问题是各类边界条件下矩形薄板问题中最难求解的情况之一。给出了不同长宽比和不同载荷比情况下四角点支承四边自由矩形薄板屈曲问题的算例，结果表明，无论是屈曲载荷还是相应的屈曲模态，本文方法得到的结果均与精细有限元分析结果吻合很好，从而证明了本文方法以及所得解析解的正确性。

2 矩形薄板屈曲问题的 Hamilton 体系控制方程与本征问题

利用 Hellinger-Reissner 两类变量的变分原理，并结合拉格朗日乘子法[18]，在区域 Ω 内可以由

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变分原理(1)描述薄板的屈曲问题[19]，

$$\delta\Pi_H=\delta\iint_\Omega\left\{\frac{D}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\frac{D}{2}\left(\frac{\partial\theta}{\partial y}\right)^2+T\left(\theta-\frac{\partial w}{\partial y}\right)+D\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial\theta}{\partial y}+D(1-\nu)\left(\frac{\partial\theta}{\partial x}\right)^2-\frac{D}{2(1-\nu^2)}\left(\frac{M_y}{D}+\frac{\partial\theta}{\partial y}+\nu\frac{\partial^2 w}{\partial x^2}\right)^2+\frac{1}{2}\left[N_x\left(\frac{\partial w}{\partial x}\right)^2+N_y\theta^2\right]\right\}\mathrm{d}x\mathrm{d}y=0 \tag{1}$$

式中 Π_H 为泛函，Ω 为笛卡尔坐标系下的板区域，w 为薄板的模态位移，θ 定义为 $\partial w/\partial y$，$D=Eh^3/[12(1-\nu^2)]$ 为弯曲刚度，h 为厚度，E 为弹性模量，ν 为泊松比，T 为拉格朗日乘子，$M_y=-D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)$ 为弯矩，N_x 和 N_y 为中面内力。

通过对独立变量 w,θ,T 和 M_y 进行变分，由 $\delta\Pi_H=0$ 可得

$$\begin{aligned}&\frac{\partial w}{\partial y}=\theta\\&\frac{\partial\theta}{\partial y}=-\nu\frac{\partial^2 w}{\partial x^2}-\frac{M_y}{D}\\&\frac{\partial T}{\partial y}=-D(1-\nu^2)\frac{\partial^4 w}{\partial x^4}+\nu\frac{\partial^2 M_y}{\partial x^2}+N_x\frac{\partial^2 w}{\partial x^2}\\&\frac{\partial M_y}{\partial y}=-T+2D(1-\nu)\frac{\partial^2\theta}{\partial x^2}-N_y\theta\end{aligned} \tag{2}$$

式(2)可表示为
$$\frac{\partial\mathbf{Z}}{\partial y}=\mathbf{HZ} \tag{3}$$

式中 $\mathbf{Z}=[w,\ \theta,\ T,\ M_y]^{\mathrm{T}}$，$\mathbf{H}=\begin{bmatrix}\mathbf{F}&\mathbf{G}\\\mathbf{Q}&-\mathbf{F}^{\mathrm{T}}\end{bmatrix}$，在 $\mathbf{H}$ 中，$\mathbf{F}=\begin{bmatrix}0&1\\-\nu\partial^2/\partial x^2&0\end{bmatrix}$，$\mathbf{G}=\begin{bmatrix}0&0\\0&-1/D\end{bmatrix}$，

$$\mathbf{Q}=\begin{bmatrix}-\frac{D(1-\nu^2)\partial^4}{\partial x^4}+\frac{N_x\partial^2}{\partial x^2}&0\\0&\frac{2D(1-\nu)\partial^2}{\partial x^2}-N_y\end{bmatrix}。$$

可知 $T=-V_y$，其中 V_y 为等效剪力，可表示为

$$V_y=Q_y+\frac{\partial M_{xy}}{\partial x}+N_y\frac{\partial w}{\partial y} \tag{4}$$

式中 $Q_y=-D\frac{\partial}{\partial y}\nabla^2 w$ 为横向剪力，$M_{xy}=-D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}$ 为扭矩。

矩阵 $\mathbf{H}$ 为 Hamilton 算子矩阵，$\mathbf{H}$ 具有性质：$\mathbf{H}^{\mathrm{T}}=\mathbf{JHJ}$，其中 $\mathbf{J}=\begin{bmatrix}0&\mathbf{I}_2\\-\mathbf{I}_2&0\end{bmatrix}$，称为辛矩阵[9]。利用分离变量法，令 $\mathbf{Z}=\mathbf{X}(x)Y(y)$，其中 $\mathbf{X}(x)$ 为向量，可以表示为 $[w(x),\theta(x),T(x),M_y(x)]^{\mathrm{T}}$，$Y(y)$ 是关于 y 的一元函数。代入式(3)可得

$$\mathrm{d}Y(y)/\mathrm{d}y=\mu Y(y),\quad\mathbf{HX}(x)=\mu\mathbf{X}(x) \tag{5,6}$$

式中 μ 和 $\mathbf{X}(x)$ 分别是哈密顿矩阵的特征值和特征向量。与式(6)相应的特征方程为

$$\begin{vmatrix}-\mu&1&0&0\\-\nu\lambda^2&-\mu&0&-\frac{1}{D}\\N_x\lambda^2-D\lambda^4(1-\nu^2)&0&-\mu&\nu\lambda^2\\0&-N_y+2D\lambda^2(1-\nu)&-1&-\mu\end{vmatrix}=0 \tag{7}$$

展开得
$$D(\lambda^2+\mu^2)^2=N_x\lambda^2+N_y\mu^2 \tag{8}$$

解得
$$\lambda_{1,2}=\pm a_1 i,\quad\lambda_{3,4}=\pm a_2 i \tag{9}$$

式中
$$a_1=\sqrt{\mu^2-\frac{N_x}{2D}+\frac{\sqrt{N_x^2+4D\mu^2(N_y-N_x)}}{2D}}$$
$$a_2=\sqrt{\mu^2-\frac{N_x}{2D}-\frac{\sqrt{N_x^2+4D\mu^2(N_y-N_x)}}{2D}}$$

由式(9)得式(6)的 $w(x)$ 的本征解为

$$w(x)=A\cos(a_1 x)+B\sin(a_1 x)+C\cos(a_2 x)+F\sin(a_2 x) \tag{10}$$

式中 A,B,C 和 F 为常数。

3　四角点支承四边自由矩形薄板屈曲问题的辛叠加解析解

为求解四角点支承四边自由薄板的屈曲问题(原问题)，需要通过辛方法先求解出基本子问题的解析解，再通过叠加法寻求原问题的解析解。图1

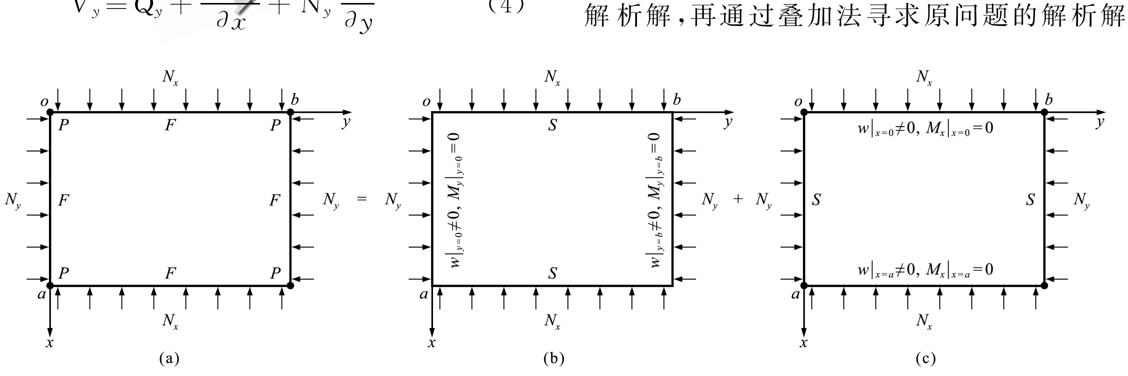


图1　辛叠加
Fig.1　Symplectic superposition

为辛叠加图解，坐标系的原点位于板的一角，板的长度为 a，宽度为 b，坐标轴 ox 和 oy 分别与板的左边和上边重合，如图 1(a)所示。原问题可以转化为两个子问题的叠加，分别如图 1(b,c)所示，其中 P 为角点支承，F 表示自由，S 表示简支。

原问题的边界条件为在四个角点上满足点支承条件，即

$$w|_{(0,0),(0,b),(a,0),(a,b)}=0 \tag{11}$$

在四条边上，满足自由边界条件，即

$$\begin{aligned} &\mathrm{V}_y|_{y=0,b}=0,\ \mathrm{M}_y|_{y=0,b}=0\\ &\mathrm{V}_x|_{x=0,a}=0,\ \mathrm{M}_x|_{x=0,a}=0 \end{aligned} \tag{12}$$

两个子问题中板的初始边界条件均为四边简支，满足边界条件：

$$\begin{aligned} &w|_{y=0,b}=0,\ \mathrm{M}_y|_{y=0,b}=0\\ &w|_{x=0,a}=0,\ \mathrm{M}_x|_{x=0,a}=0 \end{aligned} \tag{13}$$

在此基础上，对子问题(1)中板的 $y=0$ 和 $y=b$ 边分别施加 $\sum\limits_{m=1,2,3,\cdots}^{\infty}\mathrm{E}_m\sin(\alpha_m x)$ 和 $\sum\limits_{m=1,2,3,\cdots}^{\infty}\mathrm{F}_m\sin(\alpha_m x)$ 表示的位移作用；对子问题(2)中板的 $x=0$ 和 $x=a$ 边施加 $\sum\limits_{n=1,2,3,\cdots}^{\infty}\mathrm{G}_n\sin(\beta_n y)$ 和 $\sum\limits_{n=1,2,3,\cdots}^{\infty}\mathrm{H}_n\sin(\beta_n y)$ 表示的位移作用，其中 $\alpha_m=m\pi/a$，$\beta_n=n\pi/b$，E_m，F_m，G_n 和 H_n 为待定参数($m=1,2,3,\cdots;n=1,2,3,\cdots$)。

先以图 1(b)表示的子问题(1)为例进行求解。对于 $x=0$ 和 $x=a$ 边简支的矩形薄板，其在该两边的边界条件要求：

$$w(x)|_{x=0,a}=w''(x)|_{x=0,a}=0 \tag{14}$$

将式(10)代入式(14)，板发生屈曲要求常数 A，B，C 和 F 不全为 0，即要求式(14)方程组存在非零解，因此要求上述方程组的系数矩阵行列式为 0，即可得

$$\sin(aa_1)\sin(aa_2)=0 \tag{15}$$

其根为

$$a_{1,2}=\pm m\pi/a \tag{16}$$

式中 $m=1,2,3,\cdots$。于是可以解得本征值为

$$\begin{aligned} \mu_{m1,m2}&=\pm\sqrt{\frac{\mathrm{N}_y}{2\mathrm{D}}+\alpha_m^2-\frac{\sqrt{\mathrm{N}_y^2+4\mathrm{D}\alpha_m^2(\mathrm{N}_y-\mathrm{N}_x)}}{2\mathrm{D}}}\\ \mu_{m3,m4}&=\pm\sqrt{\frac{\mathrm{N}_y}{2\mathrm{D}}+\alpha_m^2+\frac{\sqrt{\mathrm{N}_y^2+4\mathrm{D}\alpha_m^2(\mathrm{N}_y-\mathrm{N}_x)}}{2\mathrm{D}}} \end{aligned} \tag{17}$$

本征向量为

$$\mathbf{X}_{mi}=\sin(\alpha_m x)\begin{Bmatrix} 1\\ \mu_{mi}\\ -\mu_{mi}\gamma_{mi}\\ \mathrm{D}(\nu\alpha_m^2-\mu_{mi}^2) \end{Bmatrix} \tag{18}$$

式中 $\gamma_{mi}=\mathrm{N}_y-\mathrm{D}[\mu_{mi}^2-\alpha_m^2(2-\nu)]$($i=1,2,3,4$)。在该子问题中，状态向量可以表达为

$$\mathbf{Z}=\sum_{m=1}^{\infty}\sum_{i=1}^{4}\mathrm{C}_{mi}e^{\mu_{mi}y}\mathbf{X}_{mi} \tag{19}$$

式中 C_{mi}($m=1,2,3,\cdots;i=1,2,3,4$)为待定常数，可以根据 y 方向的边界条件确定。原问题拆分成子问题后，需在子问题(1)的 y 方向强加位移，对应的边界条件为

$$\begin{aligned} &w|_{y=0}=\sum_{m=1,2,3,\cdots}^{\infty}\mathrm{E}_m\sin(\alpha_m x),\ \mathrm{M}_y|_{y=0}=0\\ &w|_{y=b}=\sum_{m=1,2,3,\cdots}^{\infty}\mathrm{F}_m\sin(\alpha_m x),\ \mathrm{M}_y|_{y=b}=0 \end{aligned} \tag{20}$$

将式(19)代入式(20)，即可确定子问题(1)的模态位移表达式为

$$\begin{aligned} w_1(\overline{x},\overline{y})=a\sum_{m=1,2,3,\cdots}^{\infty}&\frac{\sin(m\pi\overline{x})}{\varepsilon_m^2-\eta_m^2}\times\\ &\Big\{\mathrm{csch}(\phi\eta_m)(\varepsilon_m^2-m^2\pi^2\nu)\times\\ &\{\overline{\mathrm{F}}_m\mathrm{sh}(\phi\eta_m\overline{y})+\overline{\mathrm{E}}_m\mathrm{sh}[\phi\eta_m(1-\overline{y})]\}-\\ &\mathrm{csch}(\phi\varepsilon_m)(\eta_m^2-m^2\pi^2\nu)\times\\ &\{\overline{\mathrm{F}}_m\mathrm{sh}(\phi\varepsilon_m\overline{y})+\overline{\mathrm{E}}_m\mathrm{sh}[\phi\varepsilon_m(1-\overline{y})]\}\Big\} \end{aligned} \tag{21}$$

式中 $\overline{y}=y/b$，$\overline{x}=x/a$，$\overline{\mathrm{E}}_m=\mathrm{E}_m/a$，$\overline{\mathrm{F}}_m=\mathrm{F}_m/a$，$\phi=b/a$，$\varepsilon_m=a\mu_{m1}$，$\eta_m=a\mu_{m3}$。

对于图 1(c)表示的子问题(2)，求解过程与子问题(1)类似，对应的板的模态位移为

$$\begin{aligned} w_2(\overline{x},\overline{y})=b\sum_{n=1,2,3,\cdots}^{\infty}&\frac{\sin(n\pi\overline{y})}{\tilde{\varepsilon}_n^2-\tilde{\eta}_n^2}\times\\ &\Big\{\mathrm{csch}(\tilde{\phi}\tilde{\eta}_n)(\tilde{\varepsilon}_n^2-n^2\pi^2\nu)\times\\ &\{\mathrm{sh}(\tilde{\phi}\tilde{\eta}_n\overline{x})\overline{\mathrm{H}}_n+\mathrm{sh}[\tilde{\phi}\tilde{\eta}_n(1-\overline{x})]\overline{\mathrm{G}}_n\}-\\ &\mathrm{csch}(\tilde{\varepsilon}_n\tilde{\phi})(\tilde{\eta}_n^2-n^2\pi^2\nu)\times\\ &\{\mathrm{sh}(\tilde{\phi}\tilde{\varepsilon}_n\overline{x})\overline{\mathrm{H}}_n+\mathrm{sh}[\tilde{\phi}\tilde{\varepsilon}_n(1-\overline{x})]\overline{\mathrm{G}}_n\}\Big\} \end{aligned} \tag{22}$$

式中 $\overline{\mathrm{G}}_n=\mathrm{G}_n/b$，$\overline{\mathrm{H}}_n=\mathrm{H}_n/b$，$\tilde{\varepsilon}_n=b\mu_{n1}$

$$\tilde{\eta}_n=b\mu_{n3},\ \tilde{\phi}=a/b$$

$$\mu_{n1}=\sqrt{\frac{\mathrm{N}_x}{2\mathrm{D}}+\beta_n^2-\frac{\sqrt{\mathrm{N}_x^2+4\mathrm{D}\beta_n^2(\mathrm{N}_x-\mathrm{N}_y)}}{2\mathrm{D}}}$$

$$\mu_{n3}=\sqrt{\frac{\mathrm{N}_x}{2\mathrm{D}}+\beta_n^2+\frac{\sqrt{\mathrm{N}_x^2+4\mathrm{D}\beta_n^2(\mathrm{N}_x-\mathrm{N}_y)}}{2\mathrm{D}}}$$

通过上述推导，得到了两个子问题的模态位移解，其他各物理量，如弯矩和转角等都可以相应地导出。待定参数 E_m，F_m，G_n 和 H_n 需根据子问题的

叠加与原问题的等价性确定。根据四角点支承四边自由矩形薄板的边界条件，要求四个角点上的位移为0，子问题叠加后，无论待定参数 E_m，F_m，G_n 和 H_n 取何值，角点位移为0的条件均已满足；要求沿 $y=0$，$y=b$，$x=0$ 和 $x=a$ 四条自由边，均满足弯矩和等效剪力为0。子问题叠加后，弯矩为0的条件已经满足，只需再满足等效剪力边界条件即可。

对于 $y=0$ 边，叠加两个子问题沿 $y=0$ 边的等效剪力，令其和为0，即 $V_y|_{y=0}=\sum_{i=1}^{2}V_y^i|_{y=0}=0$，化简后得到第一组方程：

$$\begin{aligned}&\frac{1}{\varepsilon_m^2-\eta_m^2}\Big\{\eta_m(\varepsilon_m^2-m^2\pi^2\nu)\operatorname{csch}(\phi\eta_m)\times\\&[\bar{F}_m-\bar{E}_m\operatorname{ch}(\phi\eta_m)][\bar{R}-\eta_m^2+m^2\pi^2(2-\nu)]-\\&\varepsilon_m(\eta_m^2-m^2\pi^2\nu)\operatorname{csch}(\phi\varepsilon_m)[\bar{F}_m-\bar{E}_m\operatorname{ch}(\phi\varepsilon_m)]\\&[\bar{R}-\varepsilon_m^2+m^2\pi^2(2-\nu)]\Big\}+\\&\sum_{n=1,2,3,\cdots}^{\infty}\frac{2mn\pi^2}{\tilde{\varepsilon}_n^2-\tilde{\eta}_n^2}[\bar{G}_n-\cos(m\pi)\bar{H}_n]\times\\&\Big\{\frac{(\tilde{\varepsilon}_n^2-n^2\pi^2\nu)[n^2\pi^2+\phi^2\bar{R}-\tilde{\eta}_n^2(2-\nu)]}{\tilde{\eta}_n^2+m^2\pi^2\phi^2}-\\&\frac{(\tilde{\eta}_n^2-n^2\pi^2\nu)[n^2\pi^2+\phi^2\bar{R}-\tilde{\varepsilon}_n^2(2-\nu)]}{\tilde{\varepsilon}_n^2+m^2\pi^2\phi^2}\Big\}=0\end{aligned}\tag{23}$$

式中 $\bar{R}=a^2N_y/D\ (m=1,2,3,\cdots)$。

对于 $y=b$ 边，叠加两个子问题沿 $y=b$ 边的等效剪力，令其和为0，即 $V_y|_{y=b}=\sum_{i=1}^{2}V_y^i|_{y=b}=0$，得到第二组方程：

$$\begin{aligned}&\frac{1}{\varepsilon_m^2-\eta_m^2}\Big\{\eta_m(\varepsilon_m^2-m^2\pi^2\nu)\operatorname{csch}(\phi\eta_m)\times\\&[\bar{E}_m-\bar{F}_m\operatorname{ch}(\phi\eta_m)][\bar{R}-\eta_m^2+m^2\pi^2(2-\nu)]-\\&\varepsilon_m(\eta_m^2-m^2\pi^2\nu)\operatorname{csch}(\phi\varepsilon_m)\times\\&[\bar{E}_m-\bar{F}_m\operatorname{ch}(\phi\varepsilon_m)][\bar{R}-\varepsilon_m^2+m^2\pi^2(2-\nu)]\Big\}-\\&\sum_{n=1,2,3,\cdots}^{\infty}\frac{2mn\pi^2}{\tilde{\varepsilon}_n^2-\tilde{\eta}_n^2}\cos(n\pi)[\bar{G}_n-\cos(m\pi)\bar{H}_n]\times\\&\Big\{\frac{(\tilde{\varepsilon}_n^2-n^2\pi^2\nu)[n^2\pi^2+\phi^2\bar{R}-\tilde{\eta}_n^2(2-\nu)]}{\tilde{\eta}_n^2+m^2\pi^2\phi^2}-\\&\frac{(\tilde{\eta}_n^2-n^2\pi^2\nu)[n^2\pi^2+\phi^2\bar{R}-\tilde{\varepsilon}_n^2(2-\nu)]}{\tilde{\varepsilon}_n^2+m^2\pi^2\phi^2}\Big\}=0\end{aligned}\tag{24}$$

对于 $x=0$ 边，叠加两个子问题沿 $x=0$ 边的等效剪力，令其和为0，即 $V_x|_{x=0}=\sum_{i=1}^{2}V_x^i|_{x=0}=0$，得到第三组方程：

$$\begin{aligned}&\frac{1}{\tilde{\varepsilon}_n^2-\tilde{\eta}_n^2}\Big\{\tilde{\eta}_n\operatorname{csch}(\tilde{\phi}\tilde{\eta}_n)(\tilde{\varepsilon}_n^2-n^2\pi^2\nu)\times\\&[\bar{H}_n-\bar{G}_n\operatorname{ch}(\tilde{\phi}\tilde{\eta}_n)][\tilde{R}-\tilde{\eta}_n^2+n^2\pi^2(2-\nu)]-\\&\tilde{\varepsilon}_n\operatorname{csch}(\tilde{\phi}\tilde{\varepsilon}_n)(\tilde{\eta}_n^2-n^2\pi^2\nu)\times\\&[\bar{H}_n-\bar{G}_n\operatorname{ch}(\tilde{\phi}\tilde{\varepsilon}_n)][\tilde{R}-\tilde{\varepsilon}_n^2+n^2\pi^2(2-\nu)]\Big\}+\\&\sum_{m=1,2,3,\cdots}^{\infty}\frac{2nm\pi^2}{\varepsilon_m^2-\eta_m^2}[\bar{E}_m-\cos(n\pi)\bar{F}_m]\times\\&\Big\{\frac{(\varepsilon_m^2-m^2\pi^2\nu)[m^2\pi^2+\tilde{\phi}^2\tilde{R}-\eta_m^2(2-\nu)]}{\eta_m^2+n^2\pi^2\tilde{\phi}^2}-\\&\frac{(\eta_m^2-m^2\pi^2\nu)[m^2\pi^2+\tilde{\phi}^2\tilde{R}-\varepsilon_m^2(2-\nu)]}{\varepsilon_m^2+n^2\pi^2\tilde{\phi}^2}\Big\}=0\end{aligned}\tag{25}$$

式中 $\tilde{R}=b^2N_x/D\ (n=1,2,3,\cdots)$。

对于 $x=a$ 边，叠加两个子问题沿 $x=a$ 边的等效剪力，令其和为0，即 $V_x|_{x=a}=\sum_{i=1}^{2}V_x^i|_{x=a}=0$，得到第四组方程：

$$\begin{aligned}&\frac{1}{\tilde{\varepsilon}_n^2-\tilde{\eta}_n^2}\Big\{\tilde{\eta}_n\operatorname{csch}(\tilde{\phi}\tilde{\eta}_n)(\tilde{\varepsilon}_n^2-n^2\pi^2\nu)\times\\&[\bar{G}_n-\bar{H}_n\operatorname{ch}(\tilde{\phi}\tilde{\eta}_n)][\tilde{R}-\tilde{\eta}_n^2+n^2\pi^2(2-\nu)]-\\&\tilde{\varepsilon}_n\operatorname{csch}(\tilde{\phi}\tilde{\varepsilon}_n)(\tilde{\eta}_n^2-n^2\pi^2\nu)[\bar{G}_n-\bar{H}_n\operatorname{ch}(\tilde{\phi}\tilde{\varepsilon}_n)]\times\\&[\tilde{R}-\tilde{\varepsilon}_n^2+n^2\pi^2(2-\nu)]\Big\}-\\&\sum_{m=1,2,3,\cdots}^{\infty}\frac{2nm\pi^2}{\varepsilon_m^2-\eta_m^2}\cos(m\pi)[\bar{E}_m-\cos(n\pi)\bar{F}_m]\times\\&\Big\{\frac{(\varepsilon_m^2-m^2\pi^2\nu)[m^2\pi^2+\tilde{\phi}^2\tilde{R}-\eta_m^2(2-\nu)]}{\eta_m^2+n^2\pi^2\tilde{\phi}^2}-\\&\frac{(\eta_m^2-m^2\pi^2\nu)[m^2\pi^2+\tilde{\phi}^2\tilde{R}-\varepsilon_m^2(2-\nu)]}{\varepsilon_m^2+n^2\pi^2\tilde{\phi}^2}\Big\}=0\end{aligned}\tag{26}$$

式(23～26)为无穷联立方程，实际取有限项，如取 $m=1,2,3,\cdots,nt$，$n=1,2,3,\cdots,nt$。板发生屈曲要求待定系数 E_m，F_m，G_n 和 H_n 不全为0，即要求式(23～26)组成的联立方程存在非零解，因此要求上述方程组的系数矩阵行列式为0，从而给出关于屈曲载荷的方程，求解即得到屈曲载荷解。确定屈曲载荷后，即可得到上述方程组的非零解，一并代回式(21,22)并求和，即得到对应的屈曲模态。

4 算　例

表 1 和表 2 给出了四角点支承四边自由矩形薄板屈曲载荷的收敛性结果。结果表明，对于当前问题，只需取 $nt=35$，即可使所有计算结果均收敛到五位有效数字的精度，因此在实际计算中取 $nt=35$。表 3 和表 4 分别给出了长宽比为 1 和 2 的四角点支承四边自由矩形薄板的屈曲载荷，其中载荷比从 0 至 5 取整数，泊松比取 0.3。通过与精细有限元分析（采用 ABAQUS 软件中 S4R 单元，单元尺寸 $0.005a$）获得的收敛结果对比可见，所有当前结果均与有限元结果吻合得很好。表 5 给出了 $b/a=1$，$N_x/N_y=1$ 时四角点支承四边自由矩形薄板的前十阶屈曲模态，对比发现，辛叠加方法也能精确求出屈曲模态。上述算例充分证实了本文求解方法的正确性和所得解析结果的精确性。

表 1　$b/a=1$，$N_x/N_y=0$ 时板的前十阶屈曲载荷 $\overline{R}$ 收敛性结果

Tab. 1　Convergence study for the first ten buckling loads, $\overline{R}$, of the plates under $N_x/N_y=0$, with $b/a=1$

Number of series terms	Modes									
	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
5	9.0971	20.160	24.300	38.837	52.654	79.304	87.491	101.91	117.29	125.97
10	9.0970	20.159	24.286	38.835	52.623	79.222	87.441	101.88	117.16	125.79
15	9.0970	20.159	24.285	38.834	52.621	79.211	87.433	101.88	117.14	125.78
20	9.0970	20.159	24.285	38.834	52.621	79.209	87.432	101.88	117.14	125.78
25	9.0970	20.159	24.285	38.834	52.621	79.208	87.432	101.88	117.13	125.78
30	9.0970	20.159	24.285	38.834	52.620	79.208	87.432	101.88	117.13	125.78
35	**9.0970**	**20.159**	**24.285**	**38.834**	**52.620**	**79.208**	**87.431**	**101.88**	**117.13**	**125.78**
40	9.0970	20.159	24.285	38.834	52.620	79.208	87.431	101.88	117.13	125.78

表 2　$b/a=2$，$N_x/N_y=5$ 时板的前十阶屈曲载荷 $\overline{R}$ 收敛性结果

Tab. 2　Convergence study for the first ten buckling loads, $\overline{R}$, of the plates under $N_x/N_y=5$, with $b/a=2$

Number of series terms	Modes									
	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
5	0.75070	1.5925	2.4167	2.7341	3.5452	4.8522	7.6311	8.2405	8.4786	8.9497
10	0.75070	1.5924	2.4145	2.7337	3.5451	4.8444	7.6279	8.1744	8.4613	8.9497
15	0.75070	1.5924	2.4145	2.7336	3.5451	4.8432	7.6272	8.1719	8.4602	8.9497
20	0.75070	1.5924	2.4144	2.7336	3.5451	4.8431	7.6271	8.1711	8.4599	8.9497
25	0.75070	1.5924	2.4144	2.7336	3.5451	4.8430	7.6271	8.1710	8.4598	8.9497
30	0.75070	1.5924	2.4144	2.7336	3.5451	4.8430	7.6271	8.1709	8.4598	8.9497
35	**0.75070**	**1.5924**	**2.4144**	**2.7336**	**3.5451**	**4.8430**	**7.6271**	**8.1709**	**8.4597**	**8.9497**
40	0.75070	1.5924	2.4144	2.7336	3.5451	4.8430	7.6271	8.1709	8.4597	8.9497

表 3　$b/a=1$ 时不同 N_x/N_y 下板的前十阶屈曲载荷 $\overline{R}$

Tab. 3　First ten buckling loads, $\overline{R}$, of the plates under different N_x/N_y, with $b/a=1$

N_x/N_y	Methods	Modes									
		1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
0	Present	9.0970	20.159	24.285	38.834	52.620	79.208	87.431	101.88	117.13	125.78
	FEM	9.0973	20.146	24.277	38.838	52.614	79.174	87.450	101.89	117.10	125.70
1	Present	7.2941	11.587	11.587	12.036	36.575	38.365	38.365	43.689	47.469	63.347
	FEM	7.2940	11.573	11.573	12.036	36.543	38.359	38.359	43.626	47.434	63.251
2	Present	4.3105	7.4301	7.9437	10.171	19.311	22.558	28.839	34.495	37.004	39.973
	FEM	4.3106	7.4263	7.9404	10.172	19.313	22.554	28.826	34.486	37.004	39.957
3	Present	2.9456	5.4470	6.0139	9.8978	12.897	15.882	21.381	27.677	29.050	29.562
	FEM	2.9457	5.4440	6.0116	9.8981	12.899	15.880	21.371	27.667	29.058	29.551
4	Present	2.2297	4.2946	4.8309	9.6817	9.7688	12.223	16.971	21.857	22.989	23.528
	FEM	2.2298	4.2922	4.8291	9.6828	9.7690	12.221	16.963	21.864	22.979	23.520
5	Present	1.7923	3.5432	4.0343	7.7496	9.6607	9.9268	14.090	17.542	19.545	19.562
	FEM	1.7924	3.5412	4.0328	7.7504	9.6608	9.9253	14.083	17.547	19.547	19.556

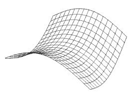
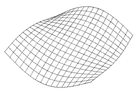
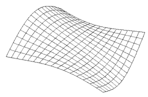
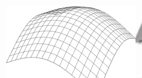
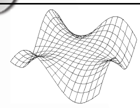
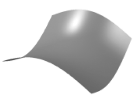
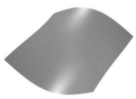
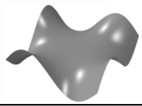
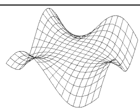
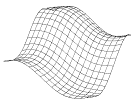
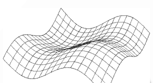
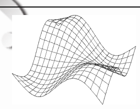
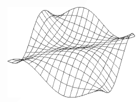
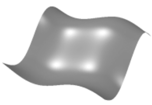
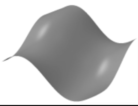
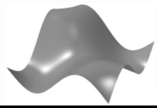
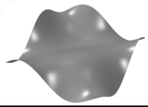
表 4　$b/a=2$ 时不同 N_x/N_y 下板的前十阶屈曲载荷 $\overline{R}$

Tab. 4　First ten buckling loads, $\overline{R}$, of the plates under different N_x/N_y, with $b/a=2$

N_x/N_y	Methods	Modes									
		1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
0	Present	2.2653	9.2262	17.988	21.192	25.206	32.226	37.504	39.617	54.254	59.767
	FEM	2.2653	9.2264	17.982	21.194	25.199	32.212	37.498	39.617	54.252	59.779
1	Present	2.2203	3.2493	8.1354	9.8110	11.665	11.903	18.884	24.056	24.207	29.280
	FEM	2.2203	3.2478	8.1350	9.8113	11.661	11.901	18.881	24.051	24.206	29.268
2	Present	1.7742	2.1461	5.0930	5.4212	7.4484	10.009	11.260	16.882	18.341	19.843
	FEM	1.7734	2.1462	5.0931	5.4205	7.4457	10.009	11.257	16.878	18.334	19.844
3	Present	1.2199	2.0184	3.6085	3.8499	5.4539	7.8099	9.6614	12.360	13.481	13.805
	FEM	1.2193	2.0184	3.6086	3.8494	5.4518	7.8075	9.6611	12.360	13.476	13.803
4	Present	0.92946	1.8208	2.9698	2.9956	4.2980	5.9751	9.2068	9.4688	10.521	10.840
	FEM	0.92902	1.8208	2.9694	2.9956	4.2963	5.9732	9.2052	9.4692	10.522	10.837
5	Present	0.75070	1.5924	2.4144	2.7336	3.5451	4.8430	7.6271	8.1709	8.4597	8.9497
	FEM	0.75035	1.5925	2.4141	2.7336	3.5437	4.8415	7.6277	8.1677	8.4601	8.9470

表 5　$b/a=1$，$N_x/N_y=1$ 时板的前十阶屈曲模态

Tab. 5　First ten buckling mode shapes of the plates under $N_x/N_y=1$ and $b/a=1$

Mode	1st	2nd	3rd	4th	5th
Present					
FEM					
Mode	6th	7th	8th	9th	10th
Present					
FEM					

5　结　论

本文采用辛叠加方法得到了单向和双向面内载荷作用下，四角点支承四边自由矩形薄板屈曲问题的新解析解。给出了不同长宽比和不同载荷比情况下板的屈曲载荷和屈曲模态的综合结果，可为其他各类近似/数值方法提供对比和检验的基准。由于辛叠加方法对解的形式不做任何假定，自始至终都是严格的解析推导，因此有望进一步推广，得到更多复杂板壳问题的新解析解。

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A new analytic solution to the buckling problem of rectangular thin plates with four corners point-supported and four edges free

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Abstract:The buckling problem of rectangular thin plates supported by corner points is an important topic in mechanics of plates and shells. However,due to the complexity of the governing equations and boundary conditions,it was difficult to obtain the analytic solutions to such problems. Although various approximate/numerical methods have been developed to solve such problems, as benchmarks, accurate analytic solutions were rarely reported in the open literature. Based on the symplectic superposition method that was proposed in recent years, the buckling problem of rectangular thin plates with four corners point-supported and four edges free is analytically solved in this paper. The problem is firstly divided into two sub-problems;then the analytic solutions of the sub-problems are derived by separation of variables and symplectic eigen expansion. The solution of the original problem is finally obtained by superposition. Since the solution procedure starts from the basic governing equation and is derived rigorously,step by step,without assuming the forms of the solutions,the presented solution method is a rational analytic method. With different aspect ratios and different in-plane load ratios,the buckling loads and typical buckling mode shapes of the rectangular thin plates with four corners point-supported and four edges free are given in the numerical examples. The correctness of the analytic solution is validated by the finite element method.

Key words:symplectic superposition method;rectangular thin plate;buckling;corner-point supports

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