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基于新修正偶应力理论的 Mindlin 层合板稳定性分析

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摘 要:基于新的各向异性修正偶应力理论提出一个Mindlin复合材料层合板稳定性模型。该理论包含纤维和基体两个不同的材料长度尺度参数。不同于忽略横向剪切应力的修正偶应力Kirchhoff薄板理论,Mindlin层合板考虑横向剪切变形引入两个转角变量。进一步建立了只含一个材料细观参数的偶应力Mindlin层合板工程理论的稳定性模型。计算了正交铺设简支方板Mindlin层合板的临界载荷。计算结果表明该模型可以用于分析细观尺度层合板稳定性的尺寸效应。

关键词:各向异性修正偶应力理论;Mindlin层合板;各向异性本构方程;稳定性;尺寸效应

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1 引 言

在细观尺度下,许多试验结果表明刚度和强度较传统理论明显增大[1-3],这种现象称作尺寸效应。设计中的变形和强度问题用传统的连续体力学理论无法解决。因此,在传统力学的基础上学者们提出了细观理论,主要包括偶应力理论和应变梯度理论。细观理论中的本构方程引入了材料长度尺度参数 l, Ji Bin 等[4]发现不同理论拟合Stolken's梁试验结果得到的细观材料参数 l 有很大差异,相差可以高达一倍,这是一个值得注意的问题。

细观偶应力理论是研究细观尺度力学行为的连续体力学理论,它已经成为连续体力学发展的一个新领域。

偶应力理论较为流行的三种类型如下。(1) Cosserat型 C^0 偶应力理论(或称偶极弹性理论, C^0 不对称偶应力理论)。(2) C^1 不对称偶应力理论。(3) C^1 修正偶应力理论。

近年来,关于偶应力理论的研究得到了很大的发展。Yang 等[5]提出修正偶应力理论,该理论中应变和曲率张量均对称。修正偶应力理论因用于工程而越来越受到关注。Park 等[6]首次按修正偶应力理论建立了 Bernoulli-Euler 梁模型,Tsiatas[7]建立了 Kirchhoff 板模型,Li 等[8]进行了 Kirchhoff 板振动分析, Ma 等[9]研究了 Mindlin 板模型,Reddy 等[10]建立了功能梯度板模型,Jung 等[11]建立了弹性介质纳米板,Shaat 等[12]分析了 Kirchhoff 纳米板弯曲问题。

陈万吉等[13-20]首次提出新的各向异性修正偶应力的 Timoshenko 梁模型。文献[16,17]建立了一系列基于新修正偶应力各向异性复合材料的梁和板的模型,其中研究了梁和板的稳定和振动问题。文献[18]基于新修正偶应力理论,建立了整体局部高阶理论的层合梁模型,首次开展层间应力的尺寸效应研究。文献[19]建立了一般性的各向异性修正偶应力理论,文献[20]基于修正偶应力理论首次开展了偶应力层合板有限元法研究并分析了偶应力层合板的尺寸效应。

本文建立了各向异性新修正偶应力 Mindlin 层合板的稳定性模型,考虑横向剪切变形,引入含转角的三个独立变量,并研究了在跨厚比为10时稳定性尺寸效应。

2 基本理论

2.1 修正偶应力理论

Yang 等[5]提出了修正偶应力理论,该理论中应变能表示为

$$U=\frac{1}{2}\int_{\Omega}(\sigma_{ij}\varepsilon_{ij}+\hat{m}_{ij}\hat{\chi}_{ij})\,\mathrm{d}v \tag{1}$$

式中

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$$\begin{cases}\varepsilon_{ij}=\dfrac{1}{2}(u_{i,j}+u_{j,i})\\ \hat{\chi}_{ij}=\dfrac{1}{2}(\omega_{i,j}+\omega_{j,i})\\ \sigma_{ij}=\lambda\varepsilon_{kk}\delta_{ij}+2G\varepsilon_{ij}\\ \hat{m}_{ij}=2Gl^2\hat{\chi}_{ij}\end{cases}\tag{2}$$

式中 u_i 为位移向量，ω_i 为转角向量，应变张量 ε_{ij} 和曲率张量 $\hat{\chi}_{ij}$ 对称，σ_{ij} 和 $\hat{m}_{ij}$ 分别为应力张量和偶应力内力矩张量，λ 和 G 为弹性常数，l 为材料长度尺度参数。

2.2 新修正偶应力理论

对传统连续体力学而言，将各向同性本构方程直接推广到各向异性比较容易，但是对修正偶应力理论而言是不可行的。传统连续体力学本构方程中的剪应变是位移导数的组合，剪应力和剪应变有对应的剪切常数，可以表达各向异性。经典各向异性偶应力理论，材料长度参数都与刚性转动对应，没有与刚性转动导数的组合的对应关系。现有的修正偶应力理论的本构方程中含刚性转动导数的组合，因此，它仅限于各向同性本构方程，不能用到各向异性。建立新的理论本构方程是关键。先按Koiter[21]提出偶应力本构模型建立各向异性本构方程，偶应力部分可表示为 $\widetilde{m}_{ij}=2l_i^2G_i\widetilde{\chi}_{ij}$，其中，$\widetilde{m}_{ij}$ 和 $\widetilde{\chi}_{ij}$（$\widetilde{\chi}_{ij}=\omega_{i,j}$）都是不对称张量。值得注意的是 l_i 与刚性转动 ω_i 对应，l_i 称为材料长度尺度参数。再按修正偶应力理论的方法将偶应力矩对称化，得到新的本构方程。将 $\widetilde{m}_{ij}=2l_i^2G_i\widetilde{\chi}_{ij}$ 代入 $m_{ij}=(\widetilde{m}_{ij}+\widetilde{m}_{ji})/2$，得到各向异性本构方程，表示为

$$\begin{cases}\sigma_{ij}=C_{ijkl}\varepsilon_{kl}\\ m_{ij}=(l_i^2G_i\chi_{ij}+l_j^2G_j\chi_{ji})\end{cases}\tag{3}$$

式中

$$\begin{cases}\varepsilon_{ij}=\dfrac{1}{2}(u_{i,j}+u_{j,i})\\ \chi_{i,j}=\omega_{i,j}\end{cases}\tag{4}$$

式中 C_{ijkl} 为弹性常数，σ_{ij}，ε_{ij} 和 m_{ij} 对称而 χ_{ij} 不对称。该理论用于各向同性则自动退化为修正偶应力理论。

3 新修正偶应力 Mindlin 层合板模型

3.1 位移场

用 u，v 和 w 表示板任意点 x，y 和 z 方向的位移，θ_y 和 θ_x 分别为绕 x 轴和 y 轴的转角。板的偶应力理论增加了 x，y 和 z 方向的转动位移 ω_x，ω_y 和 ω_z，假定与 z 无关。在整体坐标（x，y，z）下，Mindlin 层合板的位移场为

$$\begin{cases}u(x,y,z)=u_0(x,y)+z\theta_y(x,y)\\ v(x,y,z)=v_0(x,y)-z\theta_x(x,y)\\ w=w(x,y)\end{cases}\tag{5}$$

式中 u_0，v_0 和 w 为中面位移。

转动位移为

$$\begin{cases}\omega_x=\dfrac{1}{2}(w_{,y}-v_{,z})=\dfrac{1}{2}(w_{,y}+\theta_x)\\ \omega_y=\dfrac{1}{2}(u_{,z}-w_{,x})=-\dfrac{1}{2}(w_{,x}-\theta_y)\\ \omega_z=0\end{cases}\tag{6}$$

3.2 应变和曲率

按工程记法表示，Mindlin 层合板应变和曲率张量表示为

$$\varepsilon_x=u_{,x},\ \varepsilon_y=v_{,y},\ \gamma_{xy}=2\gamma_{12},\ \chi_x=\chi_{11}$$

$$\chi_y=\chi_{22},\ \chi_{xy}=\chi_{12},\ \chi_{yx}=\chi_{21}$$

由于 $\varepsilon_z=w_{,z}(x,y)=0$，$\omega_z=0$，得

$$\begin{Bmatrix}\chi_{xz}\\ \chi_{yz}\end{Bmatrix}=\begin{Bmatrix}\partial\omega_z/\partial x+\partial\omega_x/\partial z\\ \partial\omega_z/\partial y+\partial\omega_y/\partial z\end{Bmatrix}=0$$

应变和曲率表示为

$$\boldsymbol{\varepsilon}=\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{Bmatrix}=\begin{Bmatrix}\partial u/\partial x\\ \partial v/\partial y\\ \partial u/\partial y+\partial v/\partial x\\ \partial w/\partial x+\partial u/\partial z\\ \partial w/\partial y+\partial v/\partial z\end{Bmatrix}=$$

$$\begin{Bmatrix}\partial u_0/\partial x+z(\partial\theta_y/\partial x)\\ \partial v_0/\partial y-z(\partial\theta_x/\partial y)\\ \left(\dfrac{\partial u_0}{\partial y}+\dfrac{\partial v_0}{\partial x}\right)+z\left(\dfrac{\partial\theta_y}{\partial y}-\dfrac{\partial\theta_x}{\partial x}\right)\\ \partial w/\partial x+\theta_y\\ \partial w/\partial y-\theta_x\end{Bmatrix}\tag{7}$$

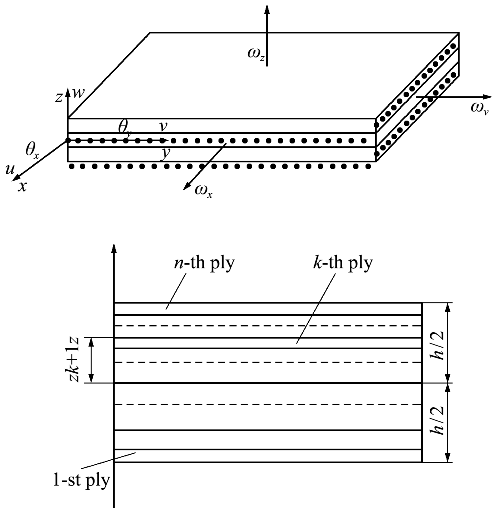


图 1 层合板示意图

Fig. 1 Schematic diagram of composite laminated plate

$$\boldsymbol{\chi}=\begin{Bmatrix}\chi_x\\ \chi_y\\ \chi_{xy}\\ \chi_{yx}\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial\omega_x}{\partial x}\\ \dfrac{\partial\omega_y}{\partial y}\\ \dfrac{\partial\omega_x}{\partial y}\\ \dfrac{\partial\omega_y}{\partial x}\end{Bmatrix}=\begin{Bmatrix}\dfrac{1}{2}\Big(\dfrac{\partial^2 w}{\partial x\partial y}+\dfrac{\partial\theta_x}{\partial x}\Big)\\ -\dfrac{1}{2}\Big(\dfrac{\partial^2 w}{\partial x\partial y}-\dfrac{\partial\theta_y}{\partial y}\Big)\\ \dfrac{1}{2}\Big(\dfrac{\partial^2 w}{\partial y^2}+\dfrac{\partial\theta_x}{\partial y}\Big)\\ -\dfrac{1}{2}\Big(\dfrac{\partial^2 w}{\partial x^2}-\dfrac{\partial\theta_y}{\partial x}\Big)\end{Bmatrix}\tag{8}$$

3.3　本构方程

第 k 层在整体坐标(x,y,z)下的本构方程表示为

$$\boldsymbol{\sigma}^k=\boldsymbol{Q}^k\boldsymbol{\varepsilon}\tag{9}$$

式中

$$\begin{cases}\boldsymbol{\sigma}^k=\big[\sigma_x^k\ \ \sigma_y^k\ \ \tau_{xy}^k\ \ \tau_{xz}^k\ \ \tau_{yz}^k\ \ m_x^k\ \ m_y^k\ \ m_{xy}^k\ \ m_{yx}^k\big]^{\mathrm{T}}\\ \boldsymbol{\varepsilon}=\big[\varepsilon_x\ \ \varepsilon_y\ \ \gamma_{xy}\ \gamma_{xz}\ \gamma_{yz}\ \chi_x\ \ \chi_y\ \ \chi_{xy}\ \chi_{yx}\big]^{\mathrm{T}}\end{cases}\tag{10}$$

$$\boldsymbol{Q}^k=\mathbf{T}^{k\mathrm{T}}\mathbf{C}^k\mathbf{T}^k\tag{11}$$

式中

$$\mathbf{T}^k=\begin{bmatrix}\mathbf{T}_1^k & \\ & \mathbf{T}_2^k\end{bmatrix}\tag{12}$$

$$\mathbf{T}_1^k=\begin{bmatrix}\cos^2\phi^k & \sin^2\phi^k & 0.5\sin 2\phi^k & & \\ \sin^2\phi^k & \cos^2\phi^k & -0.5\sin 2\phi^k & & \\ -\sin 2\phi^k & \sin 2\phi^k & \cos 2\phi^k & & \\ & & & \cos\phi^k & \sin\phi^k\\ & & & -\sin\phi^k & \cos\phi^k\end{bmatrix}\tag{13}$$

$$\mathbf{T}_2^k=\begin{bmatrix}\cos^2\phi^k & \sin^2\phi^k & 0.5\sin 2\phi^k & 0.5\sin 2\phi^k\\ \sin^2\phi^k & \cos^2\phi^k & -0.5\sin 2\phi^k & -0.5\sin 2\phi^k\\ -\sin 2\phi^k & \sin 2\phi^k & \cos^2\phi^k & -\sin^2\phi^k\\ -\sin 2\phi^k & \sin 2\phi^k & -\sin^2\phi^k & \cos^2\phi^k\end{bmatrix}\tag{14}$$

式中 ϕ^k 为铺设角。

4　新修正偶应力理论 Mindlin 层合板虚功原理

新偶应力理论 Mindlin 层合板的虚功原理为

$$\delta U-\delta W=0\tag{15}$$

式中

$$\delta U=\int_\Omega\Big(\sum_{k=1}^{n}\int_{z_k}^{z_{k+1}}(\boldsymbol{\sigma}^k)^{\mathrm{T}}\delta\boldsymbol{\varepsilon}\mathrm{d}z\Big)\mathrm{d}x\mathrm{d}y\tag{16}$$

$$\delta W=\int_\Omega\overline{\boldsymbol{f}}^{\mathrm{T}}\delta\boldsymbol{u}\,\mathrm{d}x\mathrm{d}y+\int_{\partial\Omega}\overline{\mathbf{T}}^{\mathrm{T}}\delta\boldsymbol{u}\,\mathrm{d}s\tag{17}$$

基于新修正偶应力理论,外力在虚位移上所做的虚功为

$$\delta W=\int_\Omega(f_u\delta u_0+f_v\delta v_0+f_w\delta w+f_{cx}\delta\omega_x+f_{cy}\delta\omega_y)\mathrm{d}x\mathrm{d}y+\int_{\partial\Omega}(\overline{N}_{nx}\delta u_0+\overline{N}_{ny}\delta u_0+\overline{V}\delta w+\overline{M}_n\delta\theta_n)\mathrm{d}s\tag{18}$$

$$(N_x\ N_y\ N_{xy}\ M_x\ M_y\ M_{xy}\ Q_x\ \ Q_y\ Y_x\ Y_y\ Y_{xy}\ Y_{yx})=\sum_{k=1}^{n}\int_{z_k}^{z_{k+1}}(\sigma_x^k\ \ \sigma_y^k\ \ \tau_{xy}^k\ \ z\sigma_x^k\ \ z\sigma_y^k\ \ z\tau_{xy}^k\ \ \tau_{xz}^k\ \ \tau_{yz}^k\ \ m_x^k\ \ m_y^k\ \ m_{xy}^k\ \ m_{yx}^k)\mathrm{d}z\tag{19}$$

力的边界条件为

$$\begin{cases}N_x l+N_{xy}m=\overline{N}_{nx}\\ N_{xy}l+N_y m=\overline{N}_{ny}\\ Q_x l+Q_y m-\dfrac{1}{4}\Big(\dfrac{\partial Y_x}{\partial x}-\dfrac{\partial Y_y}{\partial x}+2\dfrac{\partial Y_{xy}}{\partial y}\Big)m+\dfrac{1}{4}\Big(-\dfrac{\partial Y_x}{\partial y}+\dfrac{\partial Y_y}{\partial y}+2\dfrac{\partial Y_{yx}}{\partial x}\Big)l=\overline{V}\\ -\Big[\Big(M_y+M_x+\dfrac{1}{2}Y_{yx}-\dfrac{1}{2}Y_{xy}\Big)lm+M_{xy}+\dfrac{1}{2}(Y_y m^2-Y_x l^2)\Big]=\overline{M}_n\\ (M_x l^2-M_y m^2)+\dfrac{1}{2}Y_{xy}+\dfrac{1}{2}lm(Y_y+Y_x)=\overline{M}_s\\ Y_x-Y_y+2(Y_{xy}-Y_{yx})=0\\ Y_{yx}l^2+Y_{xy}m^2=0\end{cases}\tag{20}$$

位移边界条件为

$$\begin{cases}u_0=\overline{u}\\ v_0=\overline{v}_0\\ w=\overline{w},\ \partial w/\partial s=\partial\overline{w}/\partial s,\ \partial w/\partial n=\partial\overline{w}/\partial n\\ \theta_n=\overline{\theta}_n,\ \theta_s=\overline{\theta}_s\end{cases}\tag{21}$$

5　简化的正交铺设新修正偶应力理论 Mindlin 层合板模型

图 2 中纤维沿 x' 向铺设,l_{kb} 代表的第 k 层(x',z')面内绕纤维方向的转动细观参数,l_{km} 则是在(y',z')面内基体绕夹杂的转动细观参数。显然 $l_{kb}\gg l_{km}$,取 $l_{km}=0$,以 $l_{kb}=l_k$ 建立简化的正交铺设 Mindlin 层合板偶应力模型。

对于正交铺设板,Q^k 可以简化为

$$
Q^k=\begin{bmatrix}
Q_{11}^k & Q_{12}^k & & & & & & & \\
Q_{12}^k & Q_{22}^k & & & & & & & \\
 & & Q_{66}^k & & & & & & \\
 & & & Q_{44}^k & & & & & \\
 & & & & Q_{55}^k & & & & \\
 & & & & & 2l_k^2\tilde{Q}_{44}^k & & & \\
 & & & & & & 2l_k^2\tilde{Q}_{55}^k & & \\
 & & & & & & & l_k^2\hat{Q}_{44}^k & l_k^2\hat{Q}_{55}^k \\
 & & & & & & & l_k^2\hat{Q}_{44}^k & l_k^2\hat{Q}_{55}^k
\end{bmatrix} \tag{22}
$$

$$
\begin{cases}
Q_{11}^k = m^4 C_{11}^k + n^4 C_{22}^k \\
Q_{22}^k = n^4 C_{11}^k + m^4 C_{22}^k \\
Q_{12}^k = C_{12}^k \\
Q_{16}^k = 0 \\
Q_{26}^k = 0 \\
Q_{66}^k = C_{66}^k \\
Q_{44}^k = C_{44}^k m^2 + C_{55}^k n^2 \\
Q_{45}^k = 0 \\
Q_{55}^k = C_{44}^k n^2 + C_{55}^k m^2
\end{cases},\quad
\begin{cases}
l_k^2\hat{Q}_{44}^k = l_k^2\tilde{Q}_{44}^k = m^4 l_k^2 C_{44}^k \\
l_k^2\hat{Q}_{55}^k = l_k^2\tilde{Q}_{55}^k = n^4 l_k^2 C_{44}^k
\end{cases} \tag{23}
$$

式中 $m=\cos\phi^k,\ n=\sin\phi^k$

$$l^2=\sum_{k=1}^{n}\left[l_k^2\tilde{Q}_{ij}^k(z_{k+1}-z_k)\right]/\tilde{\tilde{Q}}_{ij}$$

并且

$$
\begin{cases}
\tilde{\tilde{Q}}_{ij}=\sum_{k=1}^{n}\left[\tilde{Q}_{ij}^k(z_{k+1}-z_k)\right] \\
\bar{Q}_{ij}=\sum_{k=1}^{n}\left[Q_{ij}^k(z_{k+1}-z_k)\right] \\
\bar{J}_{ij}=\sum_{k=1}^{n}\left[Q_{ij}^k(z_{k+1}^2-z_k^2)/2\right] \\
\bar{I}_{ij}=\sum_{k=1}^{n}\left[Q_{ij}^k(z_{k+1}^3-z_k^3)/3\right]
\end{cases} \tag{24}
$$

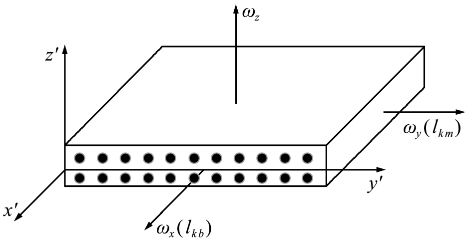


图2 复合材料第 k 层层合板细观参数(l_{kb} 和 l_{km})示意图

Fig. 2 Schematic diagram of the k layer composite laminated plate micro-material's constants (l_{kb}, l_{km})

平衡方程为

$$
\begin{cases}
\bar{Q}_{11}\dfrac{\partial^2 u_0}{\partial x^2}+\bar{Q}_{66}\dfrac{\partial^2 u_0}{\partial y^2}+(\bar{Q}_{12}+\bar{Q}_{66})\dfrac{\partial^2 v_0}{\partial x\partial y}+\bar{J}_{11}\dfrac{\partial^2\theta_y}{\partial x^2}+\bar{J}_{66}\dfrac{\partial^2\theta_y}{\partial y^2}-\bar{J}_{16}\dfrac{\partial^2\theta_x}{\partial x^2}-(\bar{J}_{12}+\bar{J}_{66})\dfrac{\partial^2\theta_x}{\partial x\partial y}+f_u=0 \\
(\bar{Q}_{66}+\bar{Q}_{12})\dfrac{\partial^2 u_0}{\partial x\partial y}+(\bar{J}_{66}+\bar{J}_{12})\dfrac{\partial^2\theta_y}{\partial x\partial y}-\bar{J}_{66}\dfrac{\partial^2\theta_x}{\partial x^2}-\bar{J}_{22}\dfrac{\partial^2\theta_x}{\partial y^2}+\bar{Q}_{66}\dfrac{\partial^2 v_0}{\partial x^2}+\bar{Q}_{22}\dfrac{\partial^2 v_0}{\partial y^2}+f_v=0 \\
\bar{Q}_{44}\dfrac{\partial^2 w}{\partial x^2}+\bar{Q}_{44}\dfrac{\partial\theta_y}{\partial x}-\dfrac{l^2}{4}\Big[(\tilde{\tilde{Q}}_{44}+\tilde{\tilde{Q}}_{55})\dfrac{\partial^4 w}{\partial x^2\partial y^2}+\tilde{\tilde{Q}}_{55}\dfrac{\partial^4 w}{\partial x^4}+\tilde{\tilde{Q}}_{44}\dfrac{\partial^4 w}{\partial y^4}+\tilde{\tilde{Q}}_{44}\dfrac{\partial^3\theta_x}{\partial x^2\partial y}+\tilde{\tilde{Q}}_{44}\dfrac{\partial^3\theta_x}{\partial y^3}-\tilde{\tilde{Q}}_{55}\dfrac{\partial^3\theta_y}{\partial x\partial y^2}-\tilde{\tilde{Q}}_{55}\dfrac{\partial^3\theta_y}{\partial x^3}\Big]+\bar{Q}_{55}\dfrac{\partial^2 w}{\partial y^2}-\bar{Q}_{55}\dfrac{\partial\theta_x}{\partial y}+\dfrac{1}{2}\Big(\dfrac{\partial f_{cy}}{\partial x}-\dfrac{\partial f_{cx}}{\partial y}\Big)+f_w=0 \\
\bar{J}_{11}\dfrac{\partial^2 u_0}{\partial x^2}+\bar{J}_{66}\dfrac{\partial^2 u_0}{\partial y^2}+(\bar{J}_{12}+\bar{J}_{66})\dfrac{\partial^2 v_0}{\partial x\partial y}+\bar{I}_{11}\dfrac{\partial^2\theta_y}{\partial x^2}+\bar{I}_{66}\dfrac{\partial^2\theta_y}{\partial y^2}-(\bar{I}_{12}+\bar{I}_{66})\dfrac{\partial^2\theta_x}{\partial x\partial y}-\bar{Q}_{44}\Big(\dfrac{\partial w}{\partial x}+\theta_y\Big)+\dfrac{l^2}{4}\Big[(-2\tilde{\tilde{Q}}_{55}+\tilde{\tilde{Q}}_{44})\dfrac{\partial^3 w}{\partial x\partial y^2}-\tilde{\tilde{Q}}_{55}\dfrac{\partial^3 w}{\partial x^3}+\tilde{\tilde{Q}}_{44}\dfrac{\partial^2\theta_x}{\partial x\partial y}+2\tilde{\tilde{Q}}_{55}\dfrac{\partial^2\theta_y}{\partial y^2}+\tilde{\tilde{Q}}_{55}\dfrac{\partial^2\theta_y}{\partial x^2}\Big]+\dfrac{1}{2}f_{cy}=0 \\
(\bar{J}_{12}+\bar{J}_{66})\dfrac{\partial^2 u_0}{\partial x\partial y}+\bar{J}_{22}\dfrac{\partial^2 v_0}{\partial y^2}+\bar{J}_{66}\dfrac{\partial^2 v_0}{\partial x^2}-\bar{I}_{66}\dfrac{\partial^2\theta_x}{\partial x^2}-\bar{I}_{22}\dfrac{\partial^2\theta_x}{\partial y^2}+(\bar{I}_{12}+\bar{I}_{66})\dfrac{\partial^2\theta_y}{\partial x\partial y}-\bar{Q}_{55}\Big(\dfrac{\partial w}{\partial y}-\theta_x\Big)+\dfrac{l^2}{4}\Big[(-2\tilde{\tilde{Q}}_{44}+\tilde{\tilde{Q}}_{55})\dfrac{\partial^3 w}{\partial x^2\partial y}-\tilde{\tilde{Q}}_{44}\dfrac{\partial^3 w}{\partial y^3}-\tilde{\tilde{Q}}_{55}\dfrac{\partial^2\theta_y}{\partial x\partial y}-2\tilde{\tilde{Q}}_{44}\dfrac{\partial^2\theta_x}{\partial x^2}-\tilde{\tilde{Q}}_{44}\dfrac{\partial^2\theta_x}{\partial y^2}\Big]-\dfrac{1}{2}f_{cx}=0
\end{cases} \tag{25}
$$

6 正交铺设新修正偶应力 Mindlin 层合板稳定分析

板受轴向力作用的稳定性分析,相对板的横向位移 w,膜向位移 u_0 和 v_0 是小量,方程(25)忽略 u_0 和 v_0,且假设 $f_u=f_v=f_{cx}=f_{cy}=0$。

如图3所示,边界条件为

$$
\begin{cases}
w|_{\Gamma}=0 \\
\partial^2 w/\partial x^2|_{x=0 \text{ or } x=L}=0 \\
\partial^2 w/\partial y^2|_{y=0 \text{ or } y=L}=0 \\
\partial\theta_x/\partial y|_{y=0 \text{ or } y=L}=0 \\
\partial\theta_y/\partial x|_{x=0 \text{ or } x=L}=0
\end{cases} \tag{26}
$$

式(25)得到位移表示的简化方程为

$$
\begin{cases}
\overline{Q}_{44}\left(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial\theta_y}{\partial x}\right)+\overline{Q}_{55}\left(\dfrac{\partial^2 w}{\partial y^2}-\dfrac{\partial\theta_x}{\partial y}\right)-\dfrac{l^2}{4}\Big[(\overset{\approx}{Q}_{44}+ \\
\overset{\approx}{Q}_{55})\dfrac{\partial^4 w}{\partial x^2\partial y^2}+\overset{\approx}{Q}_{44}\left(\dfrac{\partial^4 w}{\partial y^4}+\dfrac{\partial^3\theta_x}{\partial x^2\partial y}+\dfrac{\partial^3\theta_x}{\partial y^3}\right)+ \\
\overset{\approx}{Q}_{55}\left(\dfrac{\partial^4 w}{\partial x^4}-\dfrac{\partial^3\theta_y}{\partial x\partial y^2}-\dfrac{\partial^3\theta_y}{\partial x^3}\right)\Big]-N_{xx}\dfrac{\partial^2 w}{\partial x^2}- \\
2N_{xy}\dfrac{\partial^2 w}{\partial x\partial y}-N_{yy}\dfrac{\partial^2 w}{\partial y^2}=0 \\
-\overline{Q}_{44}\left(\dfrac{\partial w}{\partial x}+\theta_y\right)+\overline{I}_{11}\dfrac{\partial^2\theta_y}{\partial x^2}+\overline{I}_{66}\dfrac{\partial^2\theta_y}{\partial y^2}-(\overline{I}_{12}+ \\
\overline{I}_{66})\dfrac{\partial^2\theta_x}{\partial x\partial y}+\dfrac{l^2}{4}\Big[\overset{\approx}{Q}_{44}\left(\dfrac{\partial^3 w}{\partial x\partial y^2}+\dfrac{\partial^2\theta_x}{\partial x\partial y}\right)- \\
\overset{\approx}{Q}_{55}\left(2\dfrac{\partial^3 w}{\partial x\partial y^2}+\dfrac{\partial^3 w}{\partial x^3}-2\dfrac{\partial^2\theta_y}{\partial y^2}-\dfrac{\partial^2\theta_y}{\partial x^2}\right)\Big]=0 \\
-\overline{Q}_{55}\left(\dfrac{\partial w}{\partial y}-\theta_x\right)-\overline{I}_{66}\dfrac{\partial^2\theta_x}{\partial x^2}-\overline{I}_{22}\dfrac{\partial^2\theta_x}{\partial y^2}+(\overline{I}_{12}+ \\
\overline{I}_{66})\dfrac{\partial^2\theta_y}{\partial x\partial y}+\dfrac{l^2}{4}\Big[\overset{\approx}{Q}_{55}\left(\dfrac{\partial^3 w}{\partial x^2\partial y}-\dfrac{\partial^2\theta_y}{\partial x\partial y}\right)- \\
\overset{\approx}{Q}_{44}\left(2\dfrac{\partial^3 w}{\partial x^2\partial y}+\dfrac{\partial^3 w}{\partial y^3}+2\dfrac{\partial^2\theta_x}{\partial x^2}+\dfrac{\partial^2\theta_x}{\partial y^2}\right)\Big]=0
\end{cases} \tag{27}
$$

式中 N_{xx},N_{xy} 和 N_{yy} 为板的面内载荷。

满足全部边界条件位移势函数为

$$
\begin{cases}
w(x,y)=w_0\sin(a\pi/L)x\sin(b\pi/L)y \\
\theta_y=\theta_{ya}\cos(a\pi/L)x\sin(b\pi/L)y \\
\theta_x=\theta_{xa}\sin(a\pi/L)x\cos(b\pi/L)y
\end{cases} \tag{28}
$$

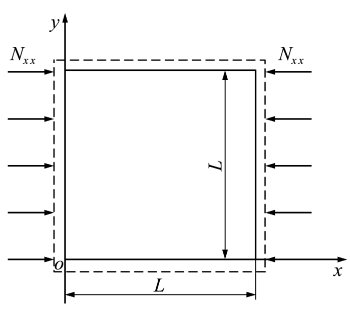


图 3　四边简支方板
Fig. 3　Simply supported laminated plate

将式(28)代入式(27)得出在单向轴压作用下稳定平衡方程为

$$
\begin{cases}
k_{11}w_0+k_{12}\theta_{ya}+k_{13}\theta_{xa}+N_{xx}(a^2\pi^2/L^2)w_0=0 \\
k_{21}w_0+k_{22}\theta_{ya}+k_{23}\theta_{xa}=0 \\
k_{31}w_0+k_{32}\theta_{ya}+k_{33}\theta_{xa}=0
\end{cases} \tag{29}
$$

由方程(29)得非零解条件下的失稳临界载荷为

$$
N_{ab}^{cr}=\frac{L^2}{a^2\pi^2}\left\{k_{11}+k_{12}\frac{k_{21}k_{33}-k_{31}k_{23}}{k_{32}k_{23}-k_{22}k_{33}}+k_{13}\frac{k_{21}k_{32}-k_{31}k_{22}}{k_{33}k_{22}-k_{23}k_{32}}\right\} \tag{30}
$$

同理,利用式(30)可以求得双向轴压作用下非零解条件下的失稳临界载荷为

$$
N_{ab}^{cr}=\frac{L^2}{(a^2+b^2)\pi^2}\left\{k_{11}+k_{12}\frac{k_{21}k_{33}-k_{31}k_{23}}{k_{32}k_{23}-k_{22}k_{33}}+k_{13}\frac{k_{21}k_{32}-k_{31}k_{22}}{k_{33}k_{22}-k_{23}k_{32}}\right\} \tag{31}
$$

式中

$$
\begin{cases}
k_{11}=-\left\{\dfrac{\pi^4 l^2}{4L^4}\left[(a^2b^2+b^4)\overset{\approx}{Q}_{44}+(a^2b^2+a^4)\overset{\approx}{Q}_{55}\right]+\dfrac{\pi^2}{L^2}(a^2\overline{Q}_{44}+b^2\overline{Q}_{55})\right\} \\
k_{12}=\dfrac{\pi}{L}\left[-a\overline{Q}_{44}+\dfrac{l^2\overset{\approx}{Q}_{55}\pi^2}{4L^2}(ab^2+a^3)\right] \\
k_{13}=\dfrac{\pi}{L}\left[b\overline{Q}_{55}-\dfrac{l^2\overset{\approx}{Q}_{44}\pi^2}{4L^2}(a^2b+b^3)\right] \\
k_{21}=\dfrac{\pi}{L}\left[-a\overline{Q}_{44}+\dfrac{ab^2l^2\pi^2}{L^3}\left(\dfrac{\overset{\approx}{Q}_{55}}{2}-\dfrac{\overset{\approx}{Q}_{44}}{4}\right)+\dfrac{l^2a^3\pi^3}{4L^3}\overset{\approx}{Q}_{55}\right] \\
k_{22}=-\overline{Q}_{44}-\dfrac{a^2\pi^2}{L^2}\left(\dfrac{l^2}{4}\overset{\approx}{Q}_{55}+\overline{I}_{11}\right)-\dfrac{b^2\pi^2}{L^2}\left(\overline{I}_{66}+\dfrac{l^2}{2}\overset{\approx}{Q}_{55}\right) \\
k_{23}=\dfrac{ab\pi^2}{L^2}\left(\overline{I}_{12}+\overline{I}_{66}-\dfrac{l^2}{4}\overset{\approx}{Q}_{44}\right) \\
k_{31}=\dfrac{b\pi}{L}\overline{Q}_{55}+\dfrac{a^2bl^2\pi^3}{4L^3}(-2\overset{\approx}{Q}_{44}+\overset{\approx}{Q}_{55})-\dfrac{b^3l^2\pi^3}{4L^3}\overset{\approx}{Q}_{44} \\
k_{32}=\dfrac{ab\pi^2}{L^2}\left(\overline{I}_{12}+\overline{I}_{66}-\dfrac{l^2}{4}\overset{\approx}{Q}_{55}\right) \\
k_{33}=-\overline{Q}_{55}-\dfrac{\pi^2}{L^2}\left[\dfrac{l^2\overset{\approx}{Q}_{44}}{4}(2a^2+b^2)+b^2\overline{I}_{22}+a^2\overline{I}_{66}\right]
\end{cases} \tag{32}
$$

当 $l=0$ 时,式(27)退化为经典正交铺设 Mindlin 层合板稳定方程。

7 算 例

算例 1 计算单向轴压作用下铺设角为[90,0,90]和[0,90,0]四边简支方板的失稳临界载荷(N_{ab}^{cr})随材料长度尺度参数 l(10^{-6} m)的变化。其中,板厚 $h=2\times10^{-5}$ m,边长 $L=10h$,材料常数[22]为 $E_2=6.9\times10^9$ Pa,$E_1=25E_2$,$G_{12}=0.5E_2$,$G_{22}=0.2E_2$,$\nu_{12}=\nu_{22}=0.25$。

由表 1 和表 2 可知,层合板以[90,0,90]铺设角铺层时,$a=2$ 和 $b=1$ 时临界载荷最小,即最容易失稳;而层合板以[0,90,0]铺层时,$a=1$ 和 $b=1$ 时临界载荷最小,说明层合板的铺设角度对尺寸效应有影响。两者失稳临界载荷均随材料长度尺度参数 l 的增大而增大,并且随着失稳临界载荷的阶数越高,增大幅度越明显。

算例 2 利用改变模型 h/l 值进一步说明尺寸效应,取 $L=10h$,材料长度尺度参数 $l=5\times10^{-6}$ m,材料常数仍与算例 1 相同。对于铺设角为[90,0,90]单向轴压及铺设角为[0,90,0]双向轴压的四边简支方板,随 h/l 变化,计算各向异性新修正偶应

表 1 铺设角为[90,0,90]四边简支方板的失稳临界载荷(N_{ab}^{cr})随材料长度尺度参数 l 的变化

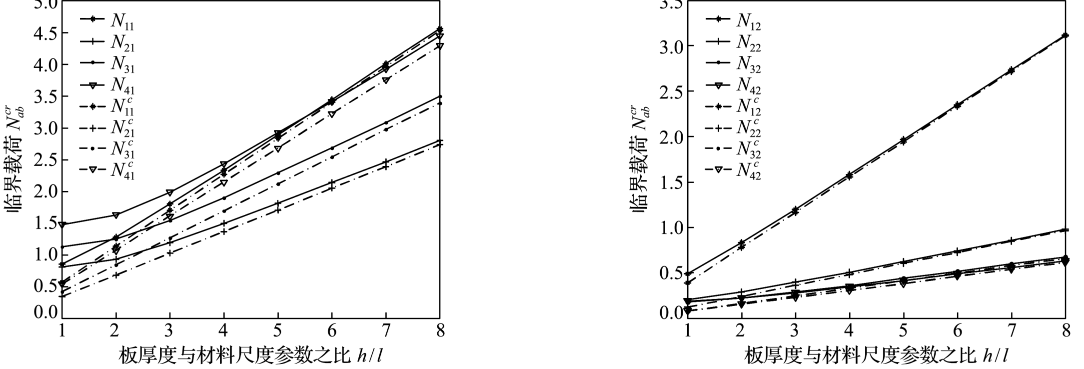
Tab. 1 Critical loads (N_{ab}^{cr}) of simply supported threelayer[90/0/90] laminated plate along with the change of material length scale parameter l

a,b	$l=0.0$	$l=0.1$	$l=0.5$	$l=1.0$	$l=1.5$	$l=3.0$	$l=6.0$
1,1	22667.8	22668.1	22675.3	22697.7	22735.1	22936.7	23739.6
2,1	13679.8	13680.3	13692.8	13731.6	13796.3	14145.0	15526.7
3,1	16991.4	16992.3	17012.2	17074.5	17178.3	17736.5	19935.4
4,1	21503.6	21504.8	21532.9	21620.9	21767.2	22553.6	25634.9
1,2	155431.0	155433.0	155459.0	155540.0	155676.0	156407.0	159305.0
2,2	48273.9	48274.9	48298.2	48370.9	48492.1	49143.7	51709.7
3,2	32970.7	32971.9	32999.9	33087.8	33233.9	34018.9	37095.5
4,2	30730.3	30731.7	30766.5	30874.9	31055.3	32023.9	35804.2

表 2 铺设角为[0,90,0]四边简支方板的失稳临界载荷(N_{ab}^{cr})随材料长度尺度参数 l 的变化

Tab. 2 Critical loads (N_{ab}^{cr}) of simply supported threelayer[0/90/0] laminated plate along with the change of material length scale parameter l

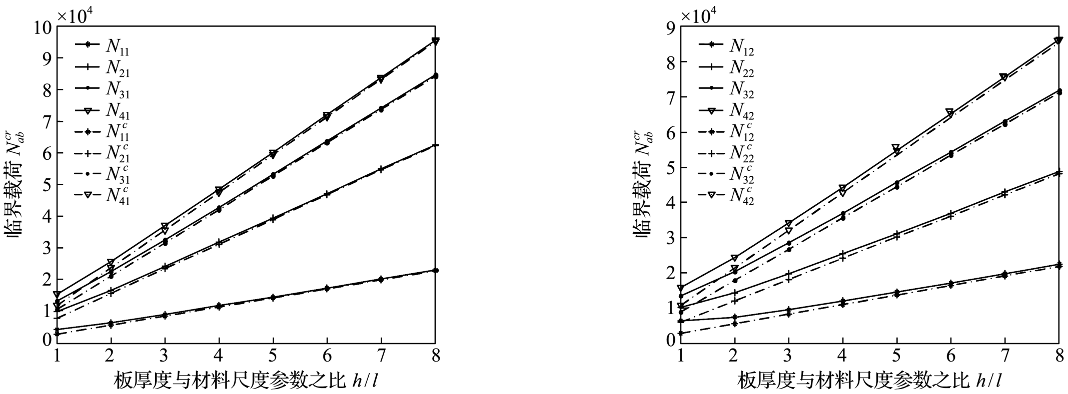
a,b	$l=0.0$	$l=0.1$	$l=0.5$	$l=1.0$	$l=1.5$	$l=3.0$	$l=6.0$
1,1	22667.8	22668.1	22675.3	22697.7	22735.1	22936.7	23739.6
2,1	38857.9	38858.1	38864.7	38885.0	38918.9	39101.7	39826.1
3,1	46681.1	46681.4	46689.3	46713.8	46754.7	46974.8	47843.2
4,1	50357.8	50358.2	50368.2	50399.4	50451.4	50731.0	51832.5
1,2	54719.2	54721.3	54771.0	54926.5	55185.3	56580.1	62106.7
2,2	48273.9	48274.9	48298.2	48370.9	48492.1	49143.7	51709.7
3,2	51449.9	51450.8	51470.4	51531.8	51633.9	52182.1	54324.4
4,2	53393.9	53394.8	53414.5	53476.2	53578.9	54129.2	56267.2



(a) a=1,2,3,4,b=1 (b) a=1,2,3,4,b=2

图 4 单向轴压铺设角为[90,0,90]四边简支方板的失稳临界载荷 N_{ab}^{cr} 随 h/l 的变化示意图

Fig. 4 Critical loads (N_{ab}^{cr}) of simply supported threelayer [90/0/90] laminated plate subjected to uniaxial compression along with the change of material length scale parameter h/l



(a) $a=1,2,3,4,b=1$　　(b) $a=1,2,3,4,b=2$

图 5　双向轴压铺设角为[0,90,0]四边简支方板的失稳临界载荷 N_{ab}^{cr} 随 h/l 的变化示意图

Fig. 5　Critical loads (N_{ab}^{cr}) of simply supported threelayer [0/90/0] laminated plate subjected to biaxial compression along with the change of material length scale parameter h/l

力理论和经典理论的临界载荷 N_{ab}^{cr}。

如图 4 和图 5 所示，N_{ab} 对应的是新修正偶应力理论的失稳临界载荷，N_{ab}^{c} 是经典理论的失稳临界载荷。可以看出，新修正偶应力理论总是大于相应各阶经典层合板理论对应的失稳临界载荷。并随着 h/l 的减小，N_{ab} 相对于 N_{ab}^{c} 越来越大；随着 h/l 的增大，N_{ab} 和 N_{ab}^{c} 的差距越来越小。这种现象表明了新修正偶应力理论 Mindlin 层合板模型能够分析层合板稳定性的尺寸效应问题。

8　结　论

本文基于新修正偶应力理论，考虑横向剪切变形，建立了 Mindlin 层合板稳定性模型，并进行尺寸效应的分析。该理论包含两个材料长度尺度参数，使其适用于分析各向异性复合材料 Mindlin 层合板。由虚功原理得到该理论下只含一个材料长度尺度参数正交铺设 Mindlin 层合简支方板的稳定性方程。算例表明，该模型的失稳临界载荷高于经典简支方板，解释了尺寸效应现象。

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Buckling analysis of composite laminated Mindlin plate based on new modified couple stress theory

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Abstract: A model of composite laminated Mindlin plate of buckling analysis based on new modified couple stress theory is presented. The theory involving two material length scale parameters, one related to fiber and the other related to matrix. Based on modified couple stress theory, the Mindlin laminated plates is different from composite laminated Kirchhoff plate, considering the transverse shear deformation, and involving two angle of rotation variables. A simplified model for cross-ply composited laminated Mindlin plate including only one material length scale parameter is applied to the buckling analysis. Calculating the critical load of cross-ply laminated Mindlin plate with simply supported boundary. Numerical results show that the scale effects of buckling analysis in microstructures are captured by present model.

Key words: anisotropic modified couple-stress theory; laminated composite Mindlin plate; anisotropic constitute relations; buckling; scale effects